

# Running Time Analysis of ACO Systems for Shortest Path Problems

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**Abstract.** Ant Colony Optimization (ACO) is inspired by the ability of ant colonies to find shortest paths between their nest and a food source. We analyze the running time of different ACO systems for shortest path problems. First, we improve running time bounds by Attiratanasunthron and Fakcharoenphol [*Information Processing Letters*, 105(3):88–92, 2008] for single-destination shortest paths and extend their results for acyclic graphs to arbitrary graphs. Our upper bound is asymptotically tight for large evaporation factors, holds with high probability, and transfers to the all-pairs shortest paths problem. There, a simple mechanism for exchanging information between ants with different destinations yields a significant improvement. Our results indicate that ACO is the best known metaheuristic for the all-pairs shortest paths problem.

## 1 Introduction

Ant Colony Optimization (ACO) is a rapidly growing field with many applications to problems from combinatorial optimization. It is inspired by the foraging behavior of real ants, which enables an ant colony to find shortest paths between its nest and a food source. Ants communicate by placing pheromone on the ground while searching the environment for food. Other ants are attracted by pheromone trails and therefore tend to follow previous ants. In case foraging ants discover different paths between a nest and a food source, a short path typically gets invested with pheromone more quickly than a longer path. The more ants take the short path, the more pheromone is deposited, until almost all ants follow the short path.

The communication mechanism of real ants has been transferred to many optimization problems such as the TSP [1], routing problems [2, 3], and many other combinatorial problems, see the book by Dorigo and Stützle [4]. Despite a plethora of applications, the theoretical knowledge on ACO is still very limited. First theoretical investigations concerned convergence proofs [5] and simplified models of ACO algorithms [6]. In 2006 the first rigorous investigations of the running time of ACO algorithms were presented independently by Gutjahr [7]

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and Neumann and Witt [8] for the optimization of simple pseudo-Boolean functions. The latter authors presented an algorithm called 1-ANT. This algorithm memorizes the best solution found so far. In each iteration a new solution is constructed and the pheromones are updated in case another solution with at least the same quality is found. In other words, every new best-so-far solution is rewarded only once. Investigations of the 1-ANT [8, 9] have shown that if the evaporation strength  $\rho$  is set too small the algorithm stagnates on even very simple problems and the expected time until an optimum is found is exponential. Other algorithms, variants of the MAX-MIN Ant System (MMAS) [10], reinforce the best-so-far solution in every iteration. This avoids the problem of stagnation and leads to efficient running times on various test problems [11, 12].

Neumann, Sudholt, and Witt [13] investigated the effect of hybridizing ACO with local search. Regarding combinatorial problems, Neumann and Witt [14] presented an analysis for minimum spanning trees. Attiratanasunthron and Fakcharoenphol [15] presented a running time analysis of ACO algorithms on a shortest path problem, the single-destination shortest path problem (SDSP) on directed acyclic graphs (DAGs). Their algorithm  $n$ -ANT is inspired both by the 1-ANT [8] and the AntNet algorithm [3]. To our knowledge, this is the first and only rigorous running time analysis for ACO on a shortest path problem. This is surprising as shortest path problems crucially inspired the development of ACO.

The aim of this work is to bring forward the theory of ACO for shortest path problems. Shortest paths have already been investigated in the context of other metaheuristics. Scharnow, Tinnefeld, and Wegener [16] presented an analysis of a simple evolutionary algorithm, the (1+1) EA, for the single-source shortest path problem (SSSP). The problems SDSP and SSSP are in essence identical. Their results were later refined by Doerr, Happ, and Klein [17]. In [18] the latter authors investigated a genetic algorithm, simply called GA, for the all-pairs shortest path problem (APSP) and proved that the use of crossover leads to a speed-up compared to mutation-based evolutionary algorithms. Finally, Horoba [19] proved that an evolutionary multiobjective algorithm represents a fully polynomial-time approximation scheme for an NP-hard multiobjective shortest path problem. Table 1 gives an overview on the best known bounds in the single-objective case, including bounds that will be proven in this paper. We remark that problem-specific algorithms solve SDSP for graphs with  $n$  vertices and  $m$  edges in time  $O(m + n \log n)$  and APSP in time  $O(nm + n^2 \log n)$  [20].

In Section 2 we define an ACO algorithm  $\text{MMAS}_{\text{SDSP}}$  for the SDSP that differs from the  $n$ -ANT [15] in two essential ways. Using our modified algorithm we are able to obtain significantly improved running time bounds (see Table 1 and Section 3) and to generalize previous results for DAGs to graphs with cycles. A corresponding lower bound shows that our upper bounds are asymptotically tight if the evaporation factor  $\rho$  is not too small. In Section 4 we transfer these results to a generalized ant system  $\text{MMAS}_{\text{APSP}}$  for the APSP where ants with different destinations move independently. The main result concerns a modification of  $\text{MMAS}_{\text{APSP}}$  where ants temporarily follow foreign pheromone traces. We prove that, surprisingly, this simple mechanism leads to a significant speed-up.

Algorithm	Problem	Iterations	Eval.
$n$ -ANT [15]	SDSP on DAGs	$O\left(\frac{m\Delta\ell\log(\Delta\ell)}{\rho}\right)$	$n$
MMAS <sub>SDSP</sub>	SDSP	$O\left(\Delta\ell\ell^* + \frac{\ell\log(\Delta\ell)}{\rho}\right)$	$n$
MMAS <sub>SDSP</sub>	SDSP on $G_{1b}$	$\Omega\left(n^2 + \frac{n}{\rho\log(1/\rho)}\right)$	$n$
MMAS <sub>SDSP</sub> +adaptive $\tau_{\min}$	SDSP	$O\left(\ell m + \frac{n\log n}{\rho}\right)$	$n$
(1+1) EA [17]	SSSP	$\Theta\left(n^2\ell^*\right)$	1
MMAS <sub>APSP</sub>	APSP	$O\left(\Delta\ell\ell^* + \frac{\ell\log(\Delta\ell)}{\rho}\right)$	$n^2$
MMAS <sub>APSP</sub> +interaction	APSP	$O\left(n\log n + \frac{\log(\ell)\log(\Delta\ell)}{\rho}\right)$	$n^2$
GA [18]	APSP	$O\left(n^{3.5}\sqrt{\log n}\right)$	1

**Table 1.** Overview on the best known running time bounds on graphs with  $n$  vertices,  $m$  edges, maximum degree  $\Delta$ , maximum number of edges  $\ell$  on any shortest path, and  $\ell^* := \max\{\ell, \ln n\}$ . The rightmost column contains the number of path length evaluations in one iteration. The bound for MMAS<sub>APSP</sub> with interaction holds for  $\rho \leq 1/(23\Delta\log n)$ ; it simplifies to  $O(n\log^3 n)$  for optimal  $\rho$ .

## 2 Algorithms

We consider shortest path problems on weighted directed graphs  $G = (V, E, w)$  where  $w(e)$  denotes the weight of edge  $e$ . The number of vertices is always denoted by  $n$ . We define a *path* of length  $\ell$  from  $u$  to  $v$  as a sequence of vertices  $(v_0, \dots, v_\ell)$  where  $v_0 = u$ ,  $v_\ell = v$ , and  $(v_{i-1}, v_i) \in E$  for all  $i$  with  $1 \leq i \leq \ell$ . For convenience, we also refer to the corresponding sequence of edges as path. Let  $\deg(u)$  denote the out-degree of a vertex  $u$  and  $\Delta(G)$  denote the maximum out-degree of any vertex  $u \in V$ . Let  $\ell(G, v) := \max_u \{\#\text{edges on } p \mid p \text{ is a shortest path from } u \text{ to } v\}$  and  $\ell(G) := \max_v \ell(G, v)$ . For undirected non-weighted graphs  $\ell(G, v)$  is called *eccentricity* of  $v$  and  $\ell(G)$  *diameter* of  $G$ .

For the single-destination shortest path problem (SDSP) we are looking for shortest paths from every vertex to a specified destination vertex. The length  $w(p)$  of a path  $p$  is defined as the sum of weights for all edges in  $p$  if the path ends with the destination vertex. If the path does not reach the destination, we define  $w(p) := \infty$ . In the following, we only consider positive weights as with negative-length cycles one can find arbitrarily short paths and the problem of computing a shortest *simple* path is NP-hard [15].

Attiratanasunthron and Fakcharoenphol [15] present the ACO algorithm  $n$ -ANT for the SDSP. Their algorithm is inspired by the 1-ANT [8] and the AntNet routing algorithm [3]. From every vertex  $u \in V$  an ant  $a_u$  starts heading for the destination. The path is chosen by performing a random walk through the graph according to pheromones on the edges. Ant  $a_u$  memorizes the best path it has found from  $u$  to the destination so far. If it has found a path that

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**Algorithm 1** Path Construction from  $u$  to  $v$  for  $\text{MMAS}_{\text{SDSP}}$ 

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1:  $i \leftarrow 0$ 
2:  $p_i \leftarrow u$ 
3:  $V_1 \leftarrow \{p \in V \setminus \{p_0\} \mid (p_0, p) \in E\}$ 
4: while  $p_i \neq v$  and  $V_{i+1} \neq \emptyset$  do
5:    $i \leftarrow i + 1$ 
6:   choose  $p_i \in V_i$  with probability  $\tau((p_{i-1}, p_i)) / \sum_{p \in V_i} \tau((p_{i-1}, p))$ 
7:    $V_{i+1} \leftarrow \{p \in V \setminus \{p_0, \dots, p_i\} \mid (p_i, p) \in E\}$ 
8: end while
9: return  $(p_0, \dots, p_i)$ 
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is at least as good as the previous best-so-far path, a pheromone update takes place and the new path is reinforced. The authors use a purely local update rule: each ant  $a_u$  is responsible for updating the edges leaving its start vertex  $u$ . If the new path is worse, the pheromones on the edges leaving  $u$  remain unchanged.

As the authors only consider acyclic graphs, the  $n$ -ANT is not supposed to deal with cycles. In particular, in [15] the authors state that in graphs with cycles their path construction procedure might take exponential time. Therefore, here we only allow ants to construct simple paths, i. e., an ant cannot visit a vertex more than once. The choice which edge to take next is made among all edges leading to unvisited vertices. This restriction bears the risk that the ant does not reach the destination. Recall that in this case the length of the path found is defined as  $w(p) = \infty$ . Due to the local pheromone update it is guaranteed that still one outgoing edge is rewarded for every vertex  $u$  with  $\deg(u) \geq 1$  and  $u \neq n$ . The construction procedure is described in Algorithm 1.

We call our algorithm  $\text{MMAS}_{\text{SDSP}}$  as we use the best-so-far update rule from the algorithm  $\text{MMAS}$  in [12] instead of the update rule used by the 1-ANT. The difference is that we always perform a pheromone update with the current best-so-far path, either with a new path or with the previous best-so-far path in case the new path is worse.

The update scheme is essentially taken over from [15]. We initialize the pheromones  $\tau: E \rightarrow \mathbb{R}_0^+$  such that all edges leaving some vertex  $u$  receive the same amount of pheromone: if  $e = (u, \cdot)$  then  $\tau(e) = 1/\deg(u)$ . If  $e$  is the only edge leaving  $u$ , we keep  $\tau(e) = 1$  fixed. This means that vertices with a single outgoing edge are traversed in the only possible way; these vertices may therefore be disregarded when proving upper bounds on the running time. In case  $u$  has more than one outgoing edge, the pheromone for  $e = (u, v)$  is computed as follows. Let  $p_u^*$  denote the best path from  $u$  found so far. Initially, we set  $p_u^*$  to an empty path, which has infinite length by definition of  $w$ . As in [10, 15] we use pheromone borders to keep pheromones within an interval  $[\tau_{\min}, \tau_{\max}]$ . In a pheromone update then

$$\tau(e) \leftarrow \begin{cases} \min \{(1 - \rho) \cdot \tau(e) + \rho, \tau_{\max}\} & \text{if } e = (u, v) \in p_u^*, \\ \max \{(1 - \rho) \cdot \tau(e), \tau_{\min}\} & \text{if } e = (u, v) \notin p_u^*. \end{cases} \quad (1)$$

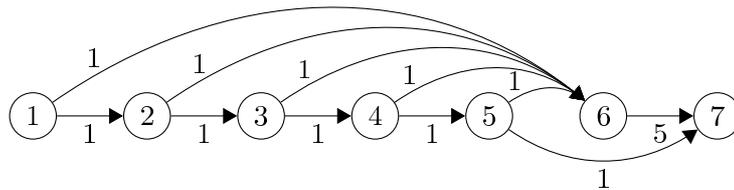
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**Algorithm 2** MMAS<sub>SDSP</sub>

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1: initialize pheromones  $\tau$  and best-so-far paths  $p_1^*, \dots, p_n^*$ 
2: loop
3:   for  $u = 1$  to  $n$  do
4:     construct a simple path  $p_u = (p_{u,0}, \dots, p_{u,\ell_u})$  from  $u$  to  $n$  w. r. t.  $\tau$ 
5:     if  $w(p_u) \leq w(p_u^*)$  then  $p_u^* \leftarrow p_u$  end if
6:   end for
7:   update pheromones  $\tau$  w. r. t.  $p_1^*, \dots, p_n^*$ 
8: end loop
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**Fig. 1.** Example graph for  $n = 7$ .

The complete algorithm is shown in Algorithm 2. We are interested in the *optimization time* or *running time* of MMAS<sub>SDSP</sub>, i. e., the number of iterations until shortest paths from  $1, \dots, n$  to  $n$  have been found. Another common performance measure for metaheuristics is the number of function evaluations. Note that in one iteration of MMAS<sub>SDSP</sub> we have  $n$  ants constructing  $n$  solutions and performing  $n$  function evaluations in parallel. Hence, the number of function evaluations is by a factor  $n$  larger than the number of iterations.

Before analyzing MMAS<sub>SDSP</sub> we motivate why it is essential to have ants starting from every vertex, even when we are only interested in the shortest path from a single source to a single destination and even when considering simple DAGs. Imagine a variant MMAS<sub>SPSP</sub> (SPSP for single-pair shortest paths) where one or multiple ants start from a single vertex, searching for the destination. Consider the following graph  $G = (V, E, w)$  sketched in Figure 1. It contains a single heavy edge  $(n-1, n)$  with weight  $n-2$  and light edges  $(u, n-1)$  for  $u \leq n-2$ ,  $(u, u+1)$  for  $u \leq n-3$  and  $(n-2, n)$  of weight 1, each.

On each vertex  $u \leq n-2$  an ant has to decide whether to move to  $n-1$  or to proceed on the shortest path. As all edges initially have equal pheromone, the probability that an ant follows the shortest path to vertex  $n/2$  is  $2^{-n/2+1}$ . Assume the ant leaves the shortest path before reaching  $n/2$ . As the length of a path containing  $\ell$  edges and traversing  $n-1$  is  $\ell + n - 3$ , no further path with a larger number of edges will be accepted in the following, except for the optimal path traversing  $1, 2, \dots, n-2, n$ . This implies that the pheromones for edges leaving the vertices  $n/2, \dots, n-2$  will always remain equal, unless an ant finds the optimum. The probability of finding the optimum is  $2^{-n/2+1}$ , hence taking the union bound over  $2^{cn}$  steps for some small constant  $c > 0$ , the optimization time is at least  $2^{cn}$  with probability  $1 - 2^{-\Omega(n)}$ . Note that this also holds in case polynomially many ants search for the destination in parallel in one iteration.

Also using edge weights as heuristic information does not help. Many ACO algorithms use both pheromones and a heuristic function to guide the solution construction [4]. However, from a vertex  $n/2 \leq u \leq n-2$  both outgoing edges have the same weight and the same pheromone, with high probability, hence they look the same for every ant. This example also shows that heuristic information may be useless or even misleading for some problem instances.

### 3 Single-Destination Shortest Path Problem

When ants start from different vertices, ants starting close to the destination have a good chance of finding a shortest path. The pheromones deposited on the outgoing edges of a vertex  $v$  can then be used to guide different ants traversing  $v$ . This way, the shortest path for  $v$  can be extended towards a longer shortest path that contains  $v$ . This is the basic idea of the analysis by Attiratanasunthron and Fakcharoenphol [15], which is improved and generalized in this section. Their results are limited to directed acyclic graphs. We start with these graphs and extend the results to directed graphs with cycles.

**Lemma 1.** *If  $\tau_{\min} + \tau_{\max} = 1$  then for every vertex  $u$  with  $\deg(u) > 1$  always*

$$1 \leq \sum_{e=(u,\cdot) \in E} \tau(e) \leq 1 + \deg(u)\tau_{\min}.$$

**Proof.** The first inequality has already been proven in [15]. Initially the sum of pheromones equals 1. Assume for an induction that  $\sum \tau(e) \geq 1$ . If the pheromones are not capped by pheromone borders, we have  $(1-\rho)\sum \tau(e) + \rho \geq 1$  as new sum. In case a pheromone drops below  $\tau_{\min}$ , setting the pheromone to  $\tau_{\min}$  can only increase the sum. If at least one pheromone is capped at the upper border  $\tau_{\max}$  then the sum of pheromones is at least  $\tau_{\min} + \tau_{\max} = 1$  as  $\deg(u) > 1$ .

For the second inequality observe that the sum of pheromones can only increase due to the lower pheromone border as  $(1-\rho)\sum \tau(e) + \rho \leq \sum \tau(e)$  follows from  $\sum \tau(e) \geq 1$ . Consider an edge  $e$  with  $(1-\rho)\tau(e) < \tau_{\min}$ . Compared to this value, the pheromone increases by at most  $\tau_{\min} \cdot \rho$  when setting the pheromone to  $\tau_{\min}$ . If currently  $\sum \tau(e) \leq 1 + \deg(u)\tau_{\min}$  then the sum of the next pheromone values is at most  $(1-\rho)(1 + \deg(u)\tau_{\min}) + \rho + \deg(u)\tau_{\min} \cdot \rho = 1 + \deg(u)\tau_{\min}$ . Hence, the second inequality follows by induction.  $\square$

As an immediate consequence, we obtain the following direct relation between pheromones and probabilities for the ant  $a_u$ , i. e., the ant starting at  $u$ , of choosing an edge  $(u, \cdot)$  in case  $\tau_{\min} \leq 1/\deg(u)$ . The last condition makes sense as  $\tau_{\min}$  should be chosen below the initial pheromone value of  $1/\deg(u)$ .

**Corollary 1.** *If  $\tau_{\min} \leq 1/\deg(u)$  and  $\tau_{\min} + \tau_{\max} = 1$  for every edge  $e = (u, \cdot)$*

$$\tau(e)/2 \leq \text{Prob}(\text{ant } a_u \text{ chooses edge } e) \leq \tau(e).$$

*The lower bound also holds for every other ant leaving vertex  $u$  and every edge  $e = (u, v)$  unless  $v$  has already been traversed by the ant. The upper bound also*

holds for every other ant and every edge  $e = (u, \cdot)$  if it has not traversed a successor of  $u$  before arriving at  $u$ .

The penultimate statement holds as the probability of choosing an edge  $e = (u, v)$  to an unvisited successor  $v$  increases if other successors of  $u$  have been visited before. In particular, we always have  $\tau_{\min}/2$  as lower bound on the probability of choosing any specific outgoing edge. This is an improvement to Lemma 1 in [15]. We remark that using the improved lemma in [15], the running time bounds for the algorithm  $n$ -ANT can be divided by  $m/n$ , where  $m$  is the number of edges.

The following theorem gives upper bounds for  $\text{MMAS}_{\text{SDSP}}$ , each consisting of two additive terms. Intuitively, the first terms cover waiting times until improvements of best-so-far paths are found. The second terms grow with  $1/\rho$ . They reflect the time to adapt the pheromones after a change of the best-so-far path. This time is called *freezing time* by Neumann, Sudholt, and Witt [12].

**Theorem 1.** *Consider a directed acyclic graph  $G$  with  $n$  vertices and positive weights. The expected optimization time of  $\text{MMAS}_{\text{SDSP}}$  on  $G$  with  $\tau_{\min} := 1/n^2$  and  $\tau_{\max} = 1 - \tau_{\min}$  is  $O(n^3 + (n \log n)/\rho)$ . Let  $\Delta := \Delta(G)$  and  $\ell := \ell(G, n)$ . The expected optimization time of  $\text{MMAS}_{\text{SDSP}}$  with  $\tau_{\min} = 1/(\Delta\ell)$  and  $\tau_{\max} = 1 - \tau_{\min}$  is  $O(n\Delta\ell + n \log(\Delta\ell)/\rho)$ .*

**Proof.** We follow the analysis by Attiratanasunthron and Fakcharoenphol [15]. Call an edge  $(u, v)$  *incorrect* if it does not belong to any shortest path from  $u$  to  $n$ . We say that a vertex  $u$  is *processed* if a shortest path from  $u$  to  $n$  has been found and if all incorrect edges leaving  $u$  have pheromone  $\tau_{\min}$ .

We estimate the expected time until a vertex  $u$  has been processed, given that all vertices reachable from  $u$  on shortest paths from  $u$  to  $n$  have already been processed. We first consider the expected time until a shortest path from  $u$  to  $n$  has been found for the first time. We also say that then vertex  $u$  has been *optimized*. By Corollary 1 the probability of choosing an edge that belongs to a shortest path from  $u$  to  $n$  is at least  $\tau_{\min}/2$ . Such a shortest path is found if the ant does not choose an incorrect edge until  $n$  is reached. As all vertices on all shortest paths are processed, all incorrect edges at some vertex  $v$  have pheromone  $\tau_{\min}$  and the probability of choosing some incorrect edge is at most  $\deg(v)\tau_{\min}$ . Hence, the probability of choosing an edge on a shortest path is at least  $1 - \deg(v)\tau_{\min} \geq 1 - 1/\ell$  if  $\tau_{\min} \leq 1/(\deg(v)\ell)$ . As all shortest paths have at most  $\ell$  edges, the probability that no incorrect edge is chosen is at least  $(1 - 1/\ell)^{\ell-1} \geq 1/e$  with  $e = \exp(1)$ . Together, the probability of finding a shortest path from  $u$  to  $n$  is at least  $\tau_{\min}/(2e)$ .

The expected time until  $u$  is optimized is thus at most  $2e/\tau_{\min}$ . Afterwards, due to the best-so-far rule, a shortest path from  $u$  to  $n$  is reinforced automatically in each iteration. The precise path may change, but it is guaranteed that only shortest paths are rewarded and hence the pheromone on incorrect edges decreases in every step. Lemma 2 in [15] states that  $\ln(\tau_{\max}/\tau_{\min})/\rho$  iterations are enough for the vertex to become processed, hence the expected time until  $u$  is processed is bounded by  $2e/\tau_{\min} + \ln(\tau_{\max}/\tau_{\min})/\rho$ .

Let  $v_1, \dots, v_{n-1}$  be an enumeration of the vertices in  $V \setminus \{n\}$  ordered with respect to increasing length of the shortest path to  $n$ . As all weights are positive, all shortest paths from  $v_{i+1}$  to  $n$  only use vertices from  $\{n, v_1, \dots, v_i\}$ . If  $v_1, \dots, v_i$  have been processed then we can wait for  $v_{i+1}$  to become processed using the above argumentation. The expected time until all vertices  $v_1, \dots, v_{n-1}$  have been processed is bounded by  $n2e/\tau_{\min} + n \ln(\tau_{\max}/\tau_{\min})/\rho$ . Choosing  $\tau_{\min} := 1/n^2$  and  $\tau_{\max} = 1 - \tau_{\min}$ , we obtain the bound  $O(n^3 + (n \log n)/\rho)$ . Choosing  $\tau_{\min} := 1/(\Delta\ell)$  and  $\tau_{\max} = 1 - \tau_{\min}$  yields the bound  $O(n\Delta\ell + n \log(\Delta\ell)/\rho)$ .  $\square$

Observe that for  $\text{MMAS}_{\text{SDSP}}$ , once a shortest path from  $u$  has been found, the pheromones are continuously “frozen” towards shortest paths from  $u$  in the following  $F = \ln(\tau_{\max}/\tau_{\min})/\rho$  iterations. The algorithm  $n$ -ANT from [15], however, only updates pheromones in case a new best-so-far path is found. This implies that a shortest path from  $u$  has to be found several times, in the worst case in  $F$  different iterations, in order to freeze the pheromones in the same way. Hence, using the best-so-far rule of MMAS algorithms leads to better performance results. This adds to the comparison of the 1-ANT and MMAS on pseudo-Boolean problems in [12].

We proceed by improving Theorem 1 in several respects. First, the bound on the expected optimization time is improved at least by a factor of  $\ell^*/n$ . Second, the result not only holds for directed acyclic graphs but for all directed graphs with positive weights and unique shortest paths. Finally, we show that the running time bounds hold with high probability (i. e. with probability at least  $1 - n^{-c}$  for some  $c > 0$ ). In the proof we follow ideas from [17] showing that the random time until a short path of length  $\ell = \Omega(\log n)$  is found is highly concentrated around the expectation.<sup>3</sup>

**Theorem 2.** *Consider a directed graph  $G$  with  $n$  vertices and positive weights where all shortest paths are unique. Let  $\Delta := \Delta(G)$ ,  $\ell := \ell(G, n)$ , and  $\ell^* := \max\{\ell, \ln n\}$ . The optimization time of  $\text{MMAS}_{\text{SDSP}}$  on  $G$  with  $\tau_{\min} = 1/(\Delta\ell)$  and  $\tau_{\max} = 1 - \tau_{\min}$  is  $O(\Delta\ell\ell^* + \ell \log(\Delta\ell)/\rho)$  with probability at least  $1 - 1/n^2$ . The optimization time bound also holds in expectation.*

**Proof.** When estimating the probability that an ant chooses an edge on a shortest path the lower bound from Corollary 1 always holds. In the proof of Theorem 1 we have shown that for ant  $a_u$  the probability of finding a shortest path from  $u$  to  $n$ , given that all successors of  $u$  on shortest paths have been processed, is bounded below by  $\tau_{\min}/(2e)$  if  $\tau_{\min} \leq 1/(\Delta\ell)$ . This result also holds in the case of arbitrary directed graphs.

Fix a vertex  $u$  and the unique shortest path  $u = v_{\ell'}, v_{\ell'-1}, \dots, v_0 = n$  with  $\ell' \leq \ell$ . We pessimistically estimate the expected time until  $u$  is processed. Let  $T_i$  be the random time until  $v_i$  is optimized. Consider random variables  $X_1, \dots, X_T$

<sup>3</sup> There is a subtle difference to [17]: in their definition of  $\ell$  the authors only consider shortest paths *with a minimum number of edges* (if there are several shortest paths between two vertices). Both definitions for  $\ell$  are, however, equal if all shortest paths are unique or have the same number of edges.

that are independently set to 1 with probability  $\tau_{\min}/(2e)$  and to 0 otherwise. The random first point of time  $T_1^*$  where  $X_t = 1$  stochastically dominates the random time until  $v_1$  is optimized. As  $v_1$  becomes processed after an additional waiting time of  $F := \ln(\tau_{\max}/\tau_{\min})/\rho$  steps,  $T_1^* + F$  stochastically dominates  $T_1$ . Inductively, we have that  $T_{\ell'}^* + \ell'F$  stochastically dominates  $T_{\ell'}$  and hence the time until  $u$  is processed.

Let  $T := 16e\ell^*/\tau_{\min}$  and  $X := \sum_{i=1}^T X_i$ . We have  $E(X) = T \cdot \tau_{\min}/(2e) = 8\ell^*$ . By Chernoff bounds [21]

$$\text{Prob}(X < \ell^*) \leq \text{Prob}(X \leq (1 - 7/8) \cdot E(X)) \leq e^{-8\ell^*(7/8)^2/2} < e^{-3\ell^*} \leq n^{-3}.$$

Hence, the probability that  $u$  is not processed after  $T + \ell \ln(\tau_{\max}/\tau_{\min})/\rho$  steps is  $1/n^3$ . By the union bound, the probability that there is an unprocessed vertex remaining after this time is at most  $1/n^2$ . The result on the expectation follows from the first result, which holds for arbitrary initial pheromones. If the algorithm does not find all shortest paths within the first  $T + \ell \ln(\tau_{\max}/\tau_{\min})/\rho$  steps, we repeat the argumentation with another phase of this length. The expected number of phases needed is clearly  $O(1)$ .  $\square$

### 3.1 Lower Bounds for MMAS<sub>SDSP</sub>

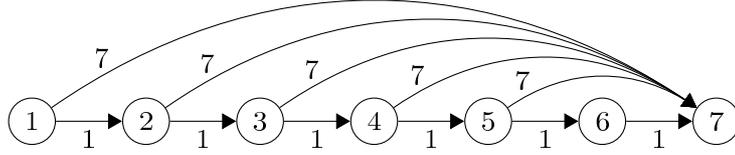
We now turn to lower bounds on the expected optimization time of MMAS<sub>SDSP</sub>. We begin with a general lower bound, which holds for a wide range of graphs, including most acyclic graphs. The main idea is that the pheromones need some time to adapt, such that a shortest path with  $\ell$  edges can be found with good probability. On the one hand, the bound grows with  $1/\rho$  if  $\rho$  is not too small. On the other hand, it also applies to pure random search, i. e.,  $\rho = 0$ .

**Theorem 3.** *Consider a directed acyclic graph  $G$  with  $n$  vertices and positive weights. Assume that  $G$  contains a unique shortest path  $p_0, \dots, p_\ell = n$  such that for  $0 \leq i < \ell$  we have  $\deg(p_i) \geq 2$  and no edges leading back from  $p_i$  to  $\{p_0, \dots, p_{i-1}\}$ . Let  $\Delta := \Delta(G)$  and  $\ell := \ell(G, n)$ . If  $\rho \leq 1 - \Omega(1)$  then the expected optimization time of MMAS<sub>SDSP</sub> on  $G$  with  $\tau_{\min} \leq 1/(\Delta\ell)$  and  $\tau_{\max} = 1 - \tau_{\min}$  is  $\Omega(\min\{(\log \ell)/\rho, e^{\sqrt{\ell}/4}\})$ .*

**Proof.** Initially all pheromones on edges  $(u, \cdot)$  equal  $1/\deg(u)$ . During the first  $t := \min\{(1/\rho - 1) \cdot \ln(\ell) \cdot 1/2, e^{\sqrt{\ell}/4}/2\} = \Omega(\min\{(\log \ell)/\rho, e^{\sqrt{\ell}/4}\})$  steps (using  $1/\rho - 1 = \Omega(1)$  by assumption on  $\rho$ ) the pheromone on every such edge is at least

$$\frac{1}{\deg(u)} \cdot (1 - \rho)^t \geq \frac{1}{\deg(u)} \cdot e^{-\ln(\ell) \cdot 1/2} = \frac{1}{\deg(u)} \cdot \frac{1}{\sqrt{\ell}}.$$

Note that this even holds in case the lower pheromone border is hit. Consider the ant starting at  $p_0$  trying to create  $p_0, \dots, p_\ell$ . As the probability of taking a specific incorrect edge is at least  $p := 1/(2 \deg(u) \sqrt{\ell})$ , the probability that the ant takes a correct edge on the path is at most  $1 - (\deg(u) - 1) \cdot p = 1 - (\deg(u) - 1) \cdot 1/(2 \deg(u) \sqrt{\ell}) \leq 1 - 1/(4\sqrt{\ell})$ . The probability that the path  $p_0, \dots, p_\ell$  is created



**Fig. 2.** Example graph  $G_{\text{lb}}$  from Definition 1 for  $n = 7$ .

in a specific iteration  $t' \leq t$  is hence bounded by  $(1 - 1/(4\sqrt{\ell}))^\ell \leq e^{-\sqrt{\ell}/4}$ . The probability that this happens during the first  $t$  iterations is bounded by  $t \cdot e^{-\sqrt{\ell}/4} \leq 1/2$  due to the definition of  $t$ . Hence with probability at least  $1/2$  we have not found all shortest paths after  $t$  steps and the lower bound  $t/2 = \Omega(\min\{(\log \ell)/\rho, e^{\sqrt{\ell}/4}\})$  follows.  $\square$

In order to assess whether the upper bound from Theorem 2 is asymptotically tight, we consider the following input instance (see Figure 2). The basic idea is that the algorithm is forced to optimize the vertices one after another, from right to left.

**Definition 1.** Let  $G_{\text{lb}} = (V, E, w)$  with  $V = \{1, \dots, n\}$ ,  $E = \{(i, i+1) \mid 1 \leq i \leq n-1\} \cup \{(i, n) \mid 1 \leq i \leq n-2\}$ , and weights  $w((u, v)) = 1$  if  $v = u+1$  and  $w((u, v)) = n$  if  $v \neq u+1$ .

Theorem 2 yields an upper bound  $O(n^2 + (n \log n)/\rho)$  for  $G_{\text{lb}}$ . The following lower bound is tight with the upper bound if  $\rho = \Omega((\log n)/n)$ . For smaller  $\rho$  there is a gap of  $O(\log^2 n)$ .

**Theorem 4.** If  $1/\text{poly}(n) \leq \rho \leq 1/2$  then the expected optimization time of  $\text{MMAS}_{\text{SDSP}}$  on  $G_{\text{lb}}$  with  $\tau_{\min} = 1/(2n)$  and  $\tau_{\max} = 1 - \tau_{\min}$  is  $\Omega\left(n^2 + \frac{n}{\rho \log(1/\rho)}\right)$ .

**Proof.** Consider all paths from  $u$  to  $n$  with  $u \leq n-2$ . The path  $(u, n)$  has length  $n$ . All other paths start with the edge  $(u, u+1)$ . The length of the path only traversing edges with weight 1 is  $n-u$ . However, if the path ends with an edge  $(v, n)$  for  $u < v \leq n-2$ , the path has length  $v-u+n > n$ . Hence the path  $(u, n)$  is the unique second best path from  $u$  to  $n$ .

Call a vertex  $u \leq n-2$  *wrong* if the best-so-far path found by ant  $a_u$  is  $(u, n)$ . After initialization both edges have an equal probability of being chosen by the first ant. By Chernoff bounds at least  $n/3$  ants  $a_u$  with  $u \leq n-2$  choose incorrect edges with probability  $1 - e^{-\Omega(n)}$  and then the edges remain incorrect until a shortest path has been found. We assume that we initially have  $n/3$  wrong vertices. First, we show that with high probability after  $F := \ln(\tau_{\max}/\tau_{\min})/\rho$  iterations we still have  $n/3 - O(\log^2 n)$  wrong vertices. For these vertices  $u$  the pheromones then are frozen towards the incorrect edge.

As long as a vertex  $u$  remains wrong, the pheromone on its correct edge is at most  $1/2$ . (It even decreases continuously towards  $\tau_{\min}$  unless a shortest path is found.) Fix the set of  $r := 8 \log(1/\rho)$  wrong vertices with largest index and let  $u$  be the vertex with the smallest index in this set. During a phase comprising the following  $t := 1/\rho - 1$  steps the probability of choosing the correct outgoing

edge is for each vertex bounded from above by  $1 - \frac{1}{4}(1 - \rho)^t \leq 1 - \frac{1}{4e}$  using Corollary 1. The probability that a shortest path for  $u$  is found throughout the phase is at most  $t(1 - \frac{1}{4e})^r \leq 2^{\log(1/\rho)}(1 - \frac{1}{4e})^{8\log(1/\rho)} \leq 1/2$ .

We conclude that the time until all  $r$  vertices have found shortest paths is at least  $t$  with probability at least  $1/2$  and the expectation is  $\Omega(t)$ . We may repeat these arguments with a new phase and another set of  $r$  vertices which are still wrong at that time and have maximal index. Consider  $3F/t = \Theta(\log n)$  subsequent phases. Applying Chernoff bounds to random variables indicating whether a phase has found shortest paths for the considered  $r$  vertices within  $t$  iterations, with high probability  $F/t$  phases each need at least  $t$  iterations. Hence, with high probability after  $F$  steps at most  $O(\log n) \cdot r = O(\log^2 n)$  wrong vertices have found shortest paths. It may happen that during a phase some vertices preceding the  $r$  considered vertices find shortest paths by chance. However, the probability that a vertex  $v$  finds a shortest path if the path still contains  $3 \log n + \log(1/\rho)$  wrong vertices is at most  $2^{-3 \log n - \log(1/\rho)} \leq \rho/n^3$ . Taking the union bound for at most  $n$  vertices and  $F$  iterations, this does not happen within  $F$  iterations, with high probability. Hence, we correct at most  $3 \log n + \log(1/\rho) = O(\log n)$  wrong vertices per phase and  $O(\log^2 n)$  wrong vertices in total this way.

With high probability we obtain a situation where for  $n/3 - O(\log^2 n)$  wrong vertices pheromones are frozen towards the incorrect edge. We separately prove lower bounds  $\Omega(n/(\rho \log(1/\rho)))$  and  $\Omega(n^2)$  for the expected remaining optimization time.

The first bound follows from applying the above arguments on phases to the remaining  $\Omega(n)$  wrong vertices, along with the fact that the probability of finding a shortest path containing  $i$  wrong vertices has decreased to  $(\tau_{\min})^i \leq 1/n^i$ . Hence, with high probability at most a constant number of wrong vertices is corrected unexpectedly per phase and the expected time to complete  $\Omega(n/r) = \Omega(n/\log(1/\rho))$  phases yields the first bound.

For the second bound  $\Omega(n^2)$  we observe that the expected time to find a shortest path for  $u$  if the path contains at least four wrong vertices is at most  $(\tau_{\min})^4 \leq 1/n^4$ . Hence, with high probability during  $\Omega(n^2)$  iterations it does not happen that more than 4 wrong vertices are corrected in the same iteration. The expected time until the wrong vertex with largest index is corrected is  $1/\tau_{\min} \geq n$ . If the number of wrong vertices always decreases by at most 4, the expected time to correct  $\Omega(n)$  wrong vertices is  $\Omega(n^2)$ .  $\square$

### 3.2 An Adaptive Choice of Pheromone Borders

The probability of constructing a shortest path from  $u$ , given that all successors of  $u$  on shortest paths have been processed, is bounded below by  $\tau_{\min}/(2e)$  if  $\tau_{\min} \leq 1/(\deg(u)\ell)$ . This suggests to choose  $\tau_{\min}$  as large as possible. However, if the same pheromone borders apply to all edges, the best feasible choice is  $\tau_{\min} = 1/(\Delta\ell)$ .

It thus makes sense to consider an ACO system where pheromone borders can be adapted to single vertices. The pheromone on an edge  $e = (u, \cdot)$  is then bounded by the pheromone borders  $\tau_{\min}(u)$  and  $\tau_{\max}(u)$ . If  $\tau_{\min}(u) =$

$1/(\deg(u)\ell)$  and  $\tau_{\max}(u) = 1 - \tau_{\min}(u)$  then the expected waiting time until  $u$  is optimized, given that all successors on shortest paths are processed, is bounded by  $2e/\tau_{\min}(u) = 2e \deg(u)\ell$ . The adaptation leads to the following bound.

**Theorem 5.** *Consider a directed graph  $G$  with  $n$  vertices,  $m$  edges, and positive weights. Let  $\ell := \ell(G, n)$ . The expected optimization time of  $\text{MMAS}_{\text{SDSP}}$  using adaptive pheromone borders with  $\tau_{\min}(u) = 1/(\deg(u)\ell)$  and  $\tau_{\max}(u) = 1 - \tau_{\min}(u)$  for all vertices  $u$  is  $O(\ell m + (n \log n)/\rho)$ .*

## 4 All-Pairs Shortest Path Problem

We now extend  $\text{MMAS}_{\text{SDSP}}$  towards an algorithm  $\text{MMAS}_{\text{APSP}}$  for the APSP. For each destination  $v \in V$  we introduce a distinct pheromone function  $\tau_v : E \rightarrow \mathbb{R}_0^+$ . In each iteration, on each vertex  $u$ , and for each destination  $v$  we have an ant  $a_{u,v}$  starting at  $u$  and heading for  $v$ . An ant heading for  $v$  uses the pheromone function  $\tau_v$  for orientation and it updates  $\tau_v$  as described in Section 2.  $\text{MMAS}_{\text{APSP}}$  remembers the best-so-far path  $p_{u,v}^*$  from  $u$  to  $v$  for all  $u, v \in V$ .

The following result is an immediate implication from Theorem 2.

**Theorem 6.** *Consider a directed graph  $G$  with  $n$  vertices and positive weights where all shortest paths are unique. Let  $\Delta := \Delta(G)$  and  $\ell := \ell(G)$ . The optimization time of  $\text{MMAS}_{\text{APSP}}$  on  $G$  with  $\tau_{\min} = 1/(\Delta\ell)$  and  $\tau_{\max} = 1 - \tau_{\min}$  is  $O(\Delta\ell^* + \ell \log(\Delta\ell)/\rho)$  with probability at least  $1 - 1/n$ . The optimization time bound also holds in expectation.*

We see that ants heading for different destinations do not collaborate in our ant system since ants heading for a destination  $v$  concern for the pheromone function  $\tau_v$  exclusively. Therefore we could also run  $n$  instances of  $\text{MMAS}_{\text{SDSP}}$  in parallel to achieve the same result. An obvious question is whether the ants can interact in some clever way to achieve a better result.

Interestingly, the following very simple mechanism proves useful. Consider the ant  $a_{u,v}$  heading for vertex  $v$ . Instead of always using the pheromone function  $\tau_v$  to travel to  $v$ , with probability, say,  $1/2$  the ant decides to follow foreign pheromones. It first chooses an intermediate destination  $w$  uniformly at random, then uses the pheromone function  $\tau_w$  to travel to  $w$ , and afterwards uses the pheromone function  $\tau_v$  to travel to the final destination  $v$  (see Algorithm 3). The pheromone update for ant  $a_{u,v}$  always applies to the pheromones  $\tau_v$ .

With this mechanism the ant  $a_{u,v}$  can profit from useful information laid down by other ants that headed towards  $w$ , in particular if  $w$  happens to be a vertex on a shortest path from  $u$  to  $v$ . The following theorem gives a significantly improved bound, without restriction to graphs with unique shortest paths.

**Theorem 7.** *Consider a directed graph  $G$  with  $n$  vertices and positive weights. Let  $\Delta := \Delta(G)$ ,  $\ell := \ell(G)$ , and  $\ell^* := \max\{\ell, \ln n\}$ . If  $\rho \leq 1/(23\Delta \log n)$  then the optimization time of  $\text{MMAS}_{\text{APSP}}$  using interaction on  $G$  with  $\tau_{\min} = 1/(\Delta\ell)$  and  $\tau_{\max} = 1 - \tau_{\min}$  is  $O(n \log n + \log(\ell) \log(\Delta\ell)/\rho)$  with probability at least  $1 - 1/n^2$ . The optimization time bound also holds in expectation.*

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**Algorithm 3** Path construction from  $u$  to  $v$  for  $\text{MMAS}_{\text{APSP}}$  with interaction

---

```
1: if getRandomBit() = 0 then
2:   construct a simple path  $p$  from  $u$  to  $v$  w. r. t.  $\tau_v$ 
3: else
4:   choose  $w \in V$  uniformly at random
5:   construct a simple path  $p' = (p'_0, \dots, p'_{\ell'})$  from  $u$  to  $w$  w. r. t.  $\tau_w$ 
6:   construct a simple path  $p'' = (p''_0, \dots, p''_{\ell''})$  from  $w$  to  $v$  w. r. t.  $\tau_v$ 
7:   if  $p'_{\ell'} = w$  then  $p \leftarrow (p'_0, \dots, p'_{\ell'}, p''_1, \dots, p''_{\ell''})$  else  $p \leftarrow p'$  end if
8: end if
9: return  $p$ 
```

---

**Proof.** We introduce similar notions as before. Consider a pair  $(u, v)$  of vertices. Let  $\ell_{u,v}$  denote the maximum number of edges of a shortest path from  $u$  to  $v$ . We call an edge *incorrect* with respect to  $v$  if it does not belong to a shortest path to  $v$ . We call  $(u, v)$  *optimized* if a shortest path from  $u$  to  $v$  has been found. We call  $(u, v)$  *processed* if it has been optimized and if the pheromone  $\tau_v(\cdot)$  on all incorrect edges  $(u, \cdot)$  is  $\tau_{\min}$ .

Consider the first  $t = (\ln 2)/\rho = O(1/\rho)$  iterations. Consider a pair  $(u, v)$  with  $\ell_{u,v} = 1$ . The probability of optimizing  $(u, v)$  in iteration  $i$  is at least  $(1 - \rho)^{i-1}/(4\Delta)$  since the ant  $a_{u,v}$  decides with probability  $1/2$  to head for  $v$  and chooses  $(u, v)$  with probability at least  $(1 - \rho)^{i-1}/(2\Delta)$  due to Corollary 1. Hence, the probability of *not* optimizing  $(u, v)$  within the considered phase is at most

$$\prod_{i=1}^t \left(1 - \frac{(1 - \rho)^{i-1}}{4\Delta}\right) \leq \exp\left(-\frac{1}{4\Delta} \sum_{i=0}^{t-1} (1 - \rho)^i\right) = \exp\left(-\frac{1 - (1 - \rho)^t}{4\Delta\rho}\right).$$

Since  $\rho \leq 1/(23\Delta \log n) \leq 1/(8\Delta \ln(2n^4))$ , the above probability is at most  $1/(2n^4)$ . Because of the union bound, all pairs  $(u, v)$  with  $\ell_{u,v} = 1$  are optimized within the considered phase with probability at least  $1 - f_1$  where  $f_1 := 1/(2n^2)$ . We know that an optimized pair  $(u, v)$  is processed within  $\ln(\tau_{\max}/\tau_{\min})/\rho$  iterations.

Consider a pair  $(u, v)$  and fix a shortest path  $p_{u,v}$  from  $u$  to  $v$  with  $\ell_{u,v}$  edges. Let  $i$  with  $(3/2)^i < \ell_{u,v} \leq (3/2)^{i+1}$ . If all pairs  $(u', v')$  with  $\ell_{u',v'} \leq (3/2)^i$  are processed, the probability of optimizing  $(u, v)$  is at least  $1/2 \cdot \ell_{u,v}/(3n) \cdot 1/e > (3/2)^i/(6en)$  since the ant decides with probability  $1/2 \cdot \ell_{u,v}/(3n)$  to choose an intermediate destination  $w$  on the middle third of  $p$ . Hence, the number of edges of all shortest paths  $p_{u,w}$  ( $p_{w,v}$ ) from  $u$  ( $w$ ) to  $w$  ( $v$ ) is at most  $(3/2)^i$ . Since  $(x, w)$  ( $(x, v)$ ) is processed for all vertices  $x$  on a shortest path from  $u$  ( $w$ ) to  $w$  ( $v$ ), the ant follows a shortest path from  $u$  to  $v$  with probability at least  $(1 - 1/\ell)^{\ell-1} \geq 1/e$ .

We divide a run of the ant system into phases. The  $i$ th phase finishes with all pairs  $(u, v)$  with  $(3/2)^{i-1} < \ell_{u,v} \leq (3/2)^i$  being processed. Since  $\ell_{u,v} \leq \ell$ , we have to consider  $\alpha := \lceil \log(\ell)/\log(3/2) \rceil$  phases.

Consider Phase  $i$  of length  $t = 6en/(3/2)^i \ln(2\alpha n^4)$ . The probability of not optimizing a pair  $(u, v)$  with  $(3/2)^{i-1} < \ell_{u,v} \leq (3/2)^i$  within the phase is at most  $(1 - (3/2)^i/(6en))^t \leq 1/(2\alpha n^4)$ . Due to the union bound, all such pairs  $(u, v)$  are optimized within  $t$  iterations with probability at least  $1 - 1/(2\alpha n^2)$ . We know that an optimized pair  $(u, v)$  is processed within  $\ln(\tau_{\max}/\tau_{\min})/\rho$  iterations. Using the union bound, all phases are finished within

$$\begin{aligned} \sum_{i=1}^{\alpha} \left( \frac{6en \ln(2\alpha n^4)}{(3/2)^i} + \frac{\ln(\Delta\ell)}{\rho} \right) &\leq 6en \ln(2\alpha n^4) \sum_{i=1}^{\alpha} \left( \frac{2}{3} \right)^i + \frac{\alpha \ln(\Delta\ell)}{\rho} \\ &= O(n \log n + \log(\ell) \log(\Delta\ell)/\rho) \end{aligned}$$

iterations with probability at least  $1 - f_2$  where  $f_2 := 1/(2n^2)$ . The first part of the theorem follows since both failure probabilities  $f_1$  and  $f_2$  sum up to  $1/n^2$ . The second part can be derived using the bound  $O(n^3 + (n \log n)/\rho)$  on the expected optimization time. This bound can be easily shown for all graphs (without restriction to unique shortest paths) using ideas from the proofs of Theorems 1 and 2.  $\square$

We remark that the choice of the probability  $1/2$  for choosing an intermediate vertex is not essential; using any other constant value  $0 < p < 1$  would only affect the constants in Theorem 7. If  $\Delta, \ell = \Omega(n)$  and  $\rho = 1/(23\Delta \log n)$  the upper bounds given in Theorem 6 and Theorem 7 simplify to  $O(n^3)$  and  $O(n \log^3 n)$ , respectively. Hence, the ant system clearly profits from our simple interaction mechanism and more collaboration between the ants.

## 5 Conclusions

ACO is motivated by the ability of real ant colonies to find shortest paths to a food source. Building on an initial study by Attiratanasunthron and Fakcharoenphol [15], we have conducted a rigorous analysis of the running time of ACO algorithms for shortest path problems. Our results (see Table 1) significantly improve and generalize the previous results for single-destination shortest paths. Taking the number of function evaluations as performance measure, the bound for MMAS<sub>SDSP</sub> is better than the bound for the (1+1) EA [17] if  $\Delta\ell = o(n)$  and  $\rho$  is not too small.

For all-pairs shortest paths first results have been obtained using MMAS<sub>APSP</sub> as a direct generalization of MMAS<sub>SDSP</sub>. We have proved that, surprisingly, letting ants temporarily follow foreign pheromone traces to random destinations yields drastically improved results. This is also the first result for combinatorial optimization where a slow adaptation for pheromones is crucial, i. e., low values for the evaporation factor  $\rho$  yield the best upper bounds. For an optimal choice of  $\rho$  the bound of  $O(n^3 \log^3 n)$  function evaluations improves upon the best known bound  $O(n^{3.5} \sqrt{\log n})$  for genetic algorithms [18]. This makes ACO the currently best known metaheuristic for the all-pairs shortest path problem from a theoretical perspective.

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