Theory of Randomised Search Heuristics
Exercise Sheet 1
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Exercise 1: A function is called unimodal if for every non-optimal search point \( x \) there is a search point \( y \) with \( f(y) > f(x) \) and \( x \) and \( y \) differ only in one bit. (Both OneMax and LeadingOnes (LO) are unimodal functions, but Jump is not.)

Consider a unimodal function \( f \) that attains integer fitness values from \( \{0, \ldots, d\} \). Use the fitness-level method to prove that the (1+1) EA optimises every such function in expected \( O(dn) \) generations.

Exercise 2: The (1+1) EA mutates its current search point by flipping every bit independently with probability 1/n. Recall from the lecture that the (1+1) EA needs \( en \ln n + O(n) \) generations on OneMax.

Now consider a (1+1) EA flipping every bit independently with a higher mutation probability of 2/n. Derive an upper bound on OneMax using the fitness-level method. Do you think this higher mutation probability is advantageous?

Hint: use \((1 - 2/n)^n \geq 1/e^2\).

Exercise 3: Let’s now consider an extension of the fitness-level method to a population-based EA. The (\( \mu + 1 \)) EA maintain a population of \( \mu \) individuals and creates one offspring in each generation:

**Algorithm 1 (\( \mu + 1 \)) EA**
1: Choose \( \mu \) individuals uniformly at random from \( \{0, 1\}^n \).
2: repeat
3: Select a parent \( x \) uniformly at random from the population.
4: Let \( y := x \). Flip each bit in \( y \) independently with probability 1/n.
5: Add \( y \) to the population.
6: Remove an individual with minimum fitness from the population.
7: until stopping criterion met

Assume a fitness-level partition \( A_0, \ldots, A_m \). Note that when the whole population is on the same fitness level, the probabilities \( s_i \) for finding better fitness levels are the same as for the (1+1) EA.

1. Determine the probability that the (1+1) EA does not flip any bit during mutation; this results in a clone of the parent. What is the expected number of generations needed to create a clone?
2. Prove the following claim: for any set \( A_i \), if the (\( \mu + 1 \)) EA has at least one individual in \( A_i \), then the expected number of generations until all individuals are in \( A_i \cup \cdots \cup A_m \) is at most \((1 - 1/n)^{-\mu} \cdot \mu \cdot H(\mu)\), where \( H(\mu) := \sum_{i=1}^{\mu} 1/i \) is the \( \mu \)-th Harmonic number.

Hints: if an individual in \( A_i \cup \cdots \cup A_m \) is selected as parent, a sufficient condition for creating an offspring in the same set is cloning the parent. The calculations leading to the \( \mu \cdot H(\mu) \) term are similar to those for coupon collecting (RLS on OneMax).

3. Recall that the fitness-level method gives an upper bound of \( \sum_{i=0}^{m-1} 1/s_i \) for the (1+1) EA. Can you give an upper bound for the (\( \mu + 1 \)) EA, with a given \( \mu \), using the statement from part 2?

4. Apply your extension of the fitness-level method to derive an upper bound for the expected optimisation time of the (\( \mu + 1 \)) EA on LeadingOnes (LO).