Exercise Sheet 1

Please send me your solutions in English or French by email by December 2, 2024.

Question 1.1 (5 marks) In the untyped λ-calculus,

(a) show that the Church numerals $c_n = \lambda s.\lambda z.\ s^n(z)$ satisfy $\mathbf{add}\ c_m c_n =_\beta c_{n+m}$, where $\mathbf{add} = \lambda x.\lambda y.\lambda s.\lambda z.\ xs(ys(z))$,

(b) show that $Y = \lambda f.\ (\lambda x.\ f(xx))\lambda x.\ f(xx)$ satisfies $M(YM) =_\beta YM$ for every λ-term M.

Question 1.2 (30 marks) In the proof of the Church-Rosser theorem for the untyped λ-calculus we have defined the relation $\to_\parallel$ on λ-terms by

- $x \to_\parallel x$, if x is a variable,
- $P \to_\parallel P'$ implies $\lambda x.\ P \to_\parallel \lambda x.\ P'$, for λ-terms P, Q,
- $P \to_\parallel P'$ and $Q \to_\parallel Q'$ imply $PQ \to_\parallel P'Q'$, for λ-terms P, P', Q, Q',
- $P \to_\parallel P'$ and $Q \to_\parallel Q'$ imply $(\lambda x.\ P)Q \to_\parallel P'[Q'/x]$, for λ-terms P, P', Q, Q'.

(a) Show that, for all λ-terms M, M', N, N',

  (i) $M \to_\parallel M$,
  (ii) $M \to_\beta N$ implies $M \to_\parallel N$,
  (iii) $M \to_\parallel N$ implies $M \to_\beta^* N$,
  (iv) $M \to_\parallel M'$ and $N \to_\parallel N'$ imply $M[N/x] \to_\parallel M'[N'/x]$.

(b) Show that $\to_\parallel$ satisfies the diamond property

$$_\parallel\!\leftarrow \cdot \to_\parallel \;\subseteq\; \to_\parallel \cdot\, _\parallel\!\leftarrow\ ,$$

where $\cdot$ indicates relational composition ($R \cdot S = \{(a,b) \mid \exists c.\ (a,c) \in R \wedge (c,b) \in S\}$ and $_\parallel\!\leftarrow\ = \to_\parallel^\circ$ indicates the relational converse of $\to_\parallel$ ($R^\circ = \{(b,a) \mid (a,b) \in R\}$).

(c) Conclude that $\to_\beta$ is confluent, that is, $_\beta^*\!\leftarrow \cdot \to_\beta^* \subseteq \to_\beta^* \cdot\, _\beta^*\!\leftarrow$.

Question 1.3 (25 marks) In the untyped λ-calculus,

(a) show that if the λ-term M has both a normal form and an infinite reduction path

$$M \to_\beta M_0 \to_\beta M_1 \to_\beta \cdots \to_\beta M_i \to_\beta \dots\ ,$$

then all the M_i on this reduction path have a normal form;

(b) show that if MN is strongly normalising then so are M and N;

(c) show that every λ-term M in normal form is of the form $\lambda y_1 \ldots \lambda y_m.\ x M_1 \ldots M_n$, where x is a variable and all M_i are in normal form.

Note that in λ-calculus, application takes precedence over abstraction, hence $\lambda x.\ PQ$ is parsed as $\lambda x.\ (PQ)$.

Question 1.4 (15 marks) Use intuitionistic natural deduction in Fitch-style to show that

(a) $\varphi \wedge (\psi \vee \chi) \vdash (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$,

(b) $\vdash ((\varphi \wedge \psi) \to \chi) \leftrightarrow (\varphi \to \psi \to \chi)$,

(c) $\vdash \phi \to \neg\neg\phi$.

Recall that $\vdash \varphi \leftrightarrow \psi$ iff $\vdash \varphi \to \psi$ and $\vdash \psi \to \varphi$, and that $\neg\varphi = \varphi \to \bot$. Search online for more information about the format of Fitch-style natural deduction beyond the examples in the lectures.

Question 1.5 (15 marks) A natural deduction calculus for classical propositional logic can be obtained by adding the *law of excluded middle* $\vdash \varphi \vee \neg\varphi$ to the intuitionistic inference rules. Show that, alternatively, the following rules of *proof by contradiction* or *double negation elimination* be added:

$$\frac{\Gamma, \neg\varphi \vdash \bot}{\Gamma \vdash \varphi}, \qquad \frac{\Gamma \vdash \neg\neg\varphi}{\Gamma \vdash \varphi}.$$

In other words: show that if either of the three rules is added, then the other two are admissible.

Question 1.6 (10 marks)

(a) Show that *Peirce's formula* $((\varphi \to \psi) \to \varphi) \to \varphi$ is derivable using natural deduction for classical propositional logic.

(b) Consider the standard (open interval) topology on $\mathbb{R}$. Interpret Peirce's formula by assigning $\mathbb{R} - \{0\}$ to φ and $\emptyset$ to ψ. Further interpret $\varphi \to \psi$ as $\mathit{int}((\mathbb{R} - I_\varphi) \cup I_\psi)$, where I_φ and I_ψ are the open intervals assigned to φ and ψ, respectively. Show that Peirce's formula does not evaluate to $\mathbb{R}$ under this interpretation.

Remark: Open sets in topological spaces under unions, intersections and implications defined as above form Heyting algebras (Boolean algebras with a weaker form of complementation), and Heyting algebras are sound and complete with respect to intuitionistic propositional logic. This means that a formula of intuitionistic propositional logic is derivable using intuitionistic natural deduction if and only if it is interpreted as the greatest element in every Heyting algebra. The example above therefore shows that Peirce's formula is not derivable in intuitionistic propositional logic.

Question 1.7 (10 marks) In the simply typed λ-calculus à la Church, find an inhabitant of the types

(a) $(\alpha \to \alpha \to \gamma) \to \alpha \to \beta \to \gamma$,

(b) $((\alpha \to \beta) \to \alpha) \to (\alpha \to \alpha \to \beta) \to \alpha$.

Question 1.8 (10 marks) Prove the subject reduction lemma in the simply typed λ-calculus à la Church: if $\Gamma \vdash M : \tau$ and $M \to_\beta^\star N$, then $\Gamma \vdash N : \tau$.

You can freely use properties such as the Thinning, Condensing, Variable, Generation or Substitution Lemma from the lectures or the literature.

Question 1.9 (40 marks) In the context of the strong normalisation proof for the simply typed λ-calculus, recall that a λ-term t *terminates*, written $t\Downarrow$, if all sequences of β-reduction with source t are finite. We have further defined a subset T of λ-terms inductively as

$$\frac{M_1, \dots, M_n \in T}{x M_1 \dots M_n \in T}, \qquad \frac{M \in T}{\lambda x.M \in T}, \qquad \frac{M[N/x]N_1 \dots N_n \in T \quad N \in T}{(\lambda x.\ M)NN_1 \dots N_N \in T}.$$

(a) Show by induction on the derivation of M that $M \in T$ implies $M\Downarrow$.

(b) Consider the two following properties:

- type τ has property (A) if for all M, N and σ such that $M : \tau \to \sigma$ and $N : \tau$, if $M, N \in T$ then $MN \in T$,
- type τ has property (B) if for all M, N, and σ such that $M : \sigma$ and $x, N : \tau$, if $M, N \in T$ then $M[N/x] \in T$.

Prove the following facts:

(i) Every type that satisfies (B) satisfies (A).

(ii) Suppose $\tau = \tau_1 \to \cdots \to \tau_n \to \tau'$, where τ' is not a function type. Then τ satisfies (B) if all τ_i satisfy (A).

(c) Show that for all λ-terms M and N, if $M, N \in T$, then $MN \in T$.

(d) Conclude as in the lectures that the simply typed λ-calculus (à la Church or Curry) is strongly normalising.

You can freely use the lemmas mentioned in Question 1.8.

Question 1.10 (15 marks) In the simply typed λ-calculus à la Church,

(a) show that $\to_\beta$ is locally confluent;

(b) use (a) to argue that $\to_\beta$ is confluent.

Again, you can freely use the facts mentioned in Question 1.8.

Question 1.11 (20 marks) In the context of the PAT principle,

(a) discuss proof normalisation for $\wedge$ and $\vee$ with respect to the β-reductions for products and coproducts (case statements);

(b) argue based on proof normalisation in intuitionistic natural deduction that the type corresponding to Peirce's formula in the simply typed λ-calculus has no inhabitant.

(c) Can GenAI answer part (b)?