Software Verification and Testing

Lecture Notes: Logic

Remember

formal methods are techniques and tools that are based on models of mathematics and mathematical logic and support the description, construction and analysis of hardware and software systems.

Why Mathematics?

mathematics offers:

- expressive formal languages, syntax, notation
- powerful methods of calculation, proof, reasoning

mathematics is:

- an abstract universal language
- an unambiguous concise tool

but:

- mathematics is often seen as difficult
- "real mathematics" is usually not formal enough
- mathematics cannot replace factual knowledge

Which Mathematics?

focus: we will consider

- propositional and predicate logic
- sets, relations, functions
- graphs and labelled transition systems

Propositional Logic

description:

- propositions are statements (of natural language) that can be either true or false
- propositions are either atomic or composed using logical connectives
- propositional logic is truth-functional: the truth of a composite proposition is a function of the truths of its components

aspects of propositional logic (PL):

- syntax: PL as a formal language
- semantics: interpretation/meaning of PL
- proof theory: formal proofs in PL
- meta-theory: issues like consistency, completeness, decidability of PL

PL Syntax

formal languages consist of an alphabet and a set of formation rules

here:

- alphabet:
  - propositional variables $p, q, \ldots$
  - logical connectives $\vee$, $\neg$
  - punctuation symbols $($, $)$
- formation rules for (propositional) formulae
  - every propositional variable is a formula
  - if ϕ is a formula, then $\neg\phi$ is a formula
  - if ϕ and ψ are formulae, then $(\phi \vee \psi)$ is a formula
  - these are all formulae

PL Syntax

remarks:

- we call the definition of formulae inductive or say that the set of formulae is closed wrt the formation rules
- we write the inductive definition more compactly (in Backus-Naur form)

$$\phi ::= p \mid \neg\phi \mid (\phi \vee \psi)$$

- propositional variables stand for atomic sentences like Elvis is alive. or It always rains in Manchester.
- $\neg$ stands for negation and $\vee$ for disjunction ("logical or").

PL Syntax

syntactic sugar: further connectives can be defined

- conjunction: $(\phi \wedge \psi)$ abbreviates $\neg(\neg\phi \vee \neg\psi)$
- implication: $(\phi \rightarrow \psi)$ abbreviates $(\neg\phi \vee \psi)$
- bi-implication: $(\phi \leftrightarrow \psi)$ abbreviates $((\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi))$

conventions: we often omit brackets and stipulate that
$\neg$ binds stronger than $\wedge, \vee$ binds stronger than $\rightarrow, \leftrightarrow$

PL Syntax

examples:

- is $\wedge p($ a formula?
- how to express She goes to bed and falls asleep.?
- how to express If p is an even number, then its successor is odd.?
- how to express Bob sends messages to the Intruder, but not to Alice.?

PL Semantics

observation: propositions can be true or false; this is their meaning

idea:

- define the meaning of a propositional variable as a function f to t (true) and f (false)
- extend f to logical connectives by truth tables

ϕ	$\neg\phi$
t	f
f	t

ϕ	ψ	$\phi \vee \psi$
t	t	t
t	f	t
f	t	t
f	f	f

PL Semantics

question: how to complete the following truth tables?

ϕ	ψ	$\phi \wedge \psi$
t	t	
t	f	
f	t	
f	f	

ϕ	ψ	$\phi \longrightarrow \psi$
t	t	
t	f	
f	t	
f	f	

ϕ	ψ	$\phi \leftrightarrow \psi$
t	t	
t	f	
f	t	
f	f	

PL Semantics

satisfaction: a meaning function f satisfies a formula ϕ,

$$f \models \phi \quad \text{iff} \quad f(\phi) = t$$

validity: a formula is valid (a tautology) iff it is satisfied by every meaning function

semantic consequence: we write $\phi \models \psi$ iff, for all meaning functions f,

$$f \models \phi \quad \text{implies} \quad f \models \psi$$

remark: we use logic informally at the meta-level for defining logic formally at the object-level. . .

PL Semantics

how to determine $f(\phi)$?

1. determine $f(p)$ for all propositional variables p occurring in ϕ
2. iteratively determine $f(\psi)$ for the subformulae ψ of ϕ (inside out), using truth tables

question: can you give a tautology and argue why it is one?

PL Proof Theory

alternative: use logical calculi to reason about valid PL formulae

logical calculi consist of

- axioms: to say what is assumed to be true
- inference rules: to construct true formulae from true ones

remark:

- there are many different kinds of logical calculi for PL
- we will present a natural deduction calculus

PL Proof Theory

definitions:

- a *judgement* ϕ true is a declaration that statement ϕ is true
- *inference rules* have the form

$$\frac{\phi_1 \text{ true} \quad \dots \quad \phi_n \text{ true}}{\psi \text{ true}}$$

  the ϕ_i true are *premises* , ψ true is the *conclusion*
- *derivations* are built from hypotheses by applying axioms and inference rules

PL Proof Theory

notation: we write

$$\begin{array}{c} \overline{\psi \ \mathsf{true}}\ u \\ \vdots \\ \phi \ \mathsf{true} \end{array}$$

to denote that the judgement ϕ true is inferred under the hypothesis ψ true

- the label u is used to make sure that assumptions are not used outside their scope
- we say that the hypothesis is *discharged* in the derivation
- all this will be clarified through examples

PL Proof Theory

inference rules come in two guises

- *introduction rules* allow us to infer judgments about a disjunction, conjunction, etc., from judgements about the disjuncts, conjuncts, etc
- *elimination rules* tell us what other judgments we can derive from a disjunction, conjunction, etc

Negation

strategy for deriving a formula $\neg\phi$: assume ϕ, derive a contradiction

introduction rule:

$$\dfrac{\dfrac{}{\phi\ \mathsf{true}}\,u \\ \vdots \\ \mathsf{F}\ \mathsf{true}}{\neg\phi\ \mathsf{true}}\,\neg_i^u$$

- if, assuming that ϕ is true, we can derive a contradiction, then $\neg\phi$ is true
- index u tells us that ϕ true must be discharged in the inference

elimination rule: any formula is true if both ϕ and $\neg\phi$ are true:

$$\dfrac{\neg\phi\ \mathsf{true} \quad \phi\ \mathsf{true}}{\mathsf{F}\ \mathsf{true}}\,\neg_e$$

Disjunction

introduction rules:

$$\frac{\phi\ \mathsf{true}}{\phi \vee \psi\ \mathsf{true}} \vee_i \qquad \frac{\psi\ \mathsf{true}}{\phi \vee \psi\ \mathsf{true}} \vee_i$$

elimination rule: (case analysis)

$$\frac{\phi \vee \psi\ \mathsf{true} \quad \begin{array}{c} \overline{\phi\ \mathsf{true}}\ u \\ \vdots \\ \chi\ \mathsf{true} \end{array} \quad \begin{array}{c} \overline{\psi\ \mathsf{true}}\ v \\ \vdots \\ \chi\ \mathsf{true} \end{array}}{\chi\ \mathsf{true}} \vee_e^{uv}$$

Conjunction

introduction/elimination rules:

$$\frac{\phi\ \mathsf{true} \qquad \psi\ \mathsf{true}}{\phi \wedge \psi\ \mathsf{true}}\wedge_i \qquad \frac{\phi \wedge \psi\ \mathsf{true}}{\phi\ \mathsf{true}}\wedge_e \qquad \frac{\phi \wedge \psi\ \mathsf{true}}{\psi\ \mathsf{true}}\wedge_e$$

Implication

introduction rule:

$$\dfrac{\begin{array}{c}\overline{\phi \text{ true}}\ u \\ \vdots \\ \psi \text{ true}\end{array}}{\phi \to \psi \text{ true}} \to_i^u$$

elimination rule: (modus ponens)

$$\dfrac{\phi \to \psi \text{ true} \quad \phi \text{ true}}{\psi \text{ true}} \to_e$$

True and False

introduction rule:

$$\frac{}{\mathsf{T}\ \mathsf{true}}\,\mathsf{T}_i$$

elimination rule: there is none for T

introduction rule: there is none for F

elimination rule:

$$\frac{\mathsf{F}\ \mathsf{true}}{\phi\ \mathsf{true}}\,\mathsf{F}_e$$

Further Rules

observation: for some proofs further rules are needed, e.g., one of

$$\frac{}{\phi \lor \neg\phi \text{ true}}\,\neg_1 \qquad \frac{\begin{array}{c}\overline{\neg\phi \text{ true}}^{\,u} \\ \vdots \\ \mathsf{F} \text{ true}\end{array}}{\phi \text{ true}}\,\neg_2 \qquad \frac{\neg\neg\phi}{\phi}\,\neg_3$$

equivalence: rules follow from implication

modus tollens: (another useful rule)

$$\frac{\phi \to \psi \text{ true} \qquad \neg\psi \text{ true}}{\neg\phi \text{ true}}$$

it can be derived (see next slide)

Examples

derivation of modus tollens:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\phi \to \psi\ \textsf{true}} \qquad \dfrac{}{\phi\ \textsf{true}}\,u}{\psi\ \textsf{true}}\,{\to_e} \qquad \dfrac{}{\neg\psi\ \textsf{true}}}{\textsf{F true}}\,\neg_e}{\neg\phi\ \textsf{true}}\,\neg_i^u$$

1. assume that $\neg\psi$ and $\phi \to \psi$ hold; show that $\neg\phi$ holds
2. assume, by reduction to absurd, that ϕ holds
3. then ψ must also hold, since ϕ and $\phi \to \psi$ imply ψ
4. but $\neg\psi$ also holds by assumption, a contradiction
5. hence $\neg\psi$ must hold by negation introduction

Examples

property: $p \rightarrow (q \rightarrow p)$ holds

proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p\ \mathsf{true}}\,u}{q\ \mathsf{true}}\,v}{p\ \mathsf{true}}}{q \rightarrow p\ \mathsf{true}}\rightarrow_i^v}{p \rightarrow (q \rightarrow p)\ \mathsf{true}}\rightarrow_i^u$$

Examples

property: $(p \vee (q \wedge p)) \rightarrow p$ holds

proof:

$$
\dfrac{\dfrac{}{p \vee (q \wedge p) \text{ true}}\,u \quad \dfrac{}{p \text{ true}}\,v \quad \dfrac{\dfrac{}{q \wedge p \text{ true}}\,w}{p \text{ true}}\,\wedge_e}{\dfrac{p \text{ true}}{(p \vee (q \wedge p)) \rightarrow p \text{ true}}\,\rightarrow_i^u}\,\vee_e^{vw}
$$

PL Meta-Theory

interesting questions:

- do inference rules really preserve validity (soundness), that is, does

$$\Gamma \vdash \phi \quad \text{imply} \quad \Gamma \models \phi \quad ?$$

- can inference rules prove all tautologies (completeness), that is, does

$$\models \phi \quad \text{imply} \quad \vdash \phi \quad ?$$

theorem: the PL calculus is sound and complete

PL Meta-Theory

further theorems:

- PL is decidable: there is an algorithm to determine satisfiability of PL formulae (truth tables yield a decision procedure, but there are more efficient ones. . .)
- the logical connectives suffice for expressing all truth functions

Predicate Logic

observation: PL cannot express some logical arguments:

all horses are animals
therefore all horses' heads are animals' heads

some girls are better than others
(therefore) some girls' mothers are better than other girls' mothers

question: why is the first argument sound, whereas the second one isn't?

Predicate Logic

is the following argument sound? (Lewis Carroll)

premises:

1. the only animals in this house are cats
2. every animal in this house is suitable for a pet,
   that loves to gaze at the moon
3. when I detest an animal, I avoid it
4. no animals are carnivorous, unless they prowl at night
5. no cat fails to kill mice
6. no animal ever take to me, except what are in this house
7. kangaroos are not suitable for pets
8. none but carnivora kill mice
9. I detest animals that do not take to me
10. animals that prowl at night always love to gaze at the moon

conclusion: I always avoid a kangaroo

Predicate Logic

task: give a more detailed formalisation of natural language

- functions: “some girls’ mother”
- predicates: “some girls are better than others”
- equality: “one plus one equals two”
- quantification: “some lectures are boring”,
  “every even number has an odd successor”

Predicate Logic

from language to logic:

1. some lectures are boring
2. something is a lecture and is boring
3. there is a thing such that it is a lecture and it is boring
4. $\exists x.isLecture(x) \wedge isBoring(x)$
5. $\exists x.L(x) \wedge B(x)$

1. every even number has an odd successor
2. every thing is such that if it is even then its successor is odd
3. $\forall x.isEven(x) \rightarrow isOdd(successor(x))$
4. $\forall x.(\exists y.x{=}2{\cdot}y) \rightarrow (\exists z.x{+}1{=}2{\cdot}z{+}1)$

Predicate Logic

notation: the predicate logic we consider is also called first-order logic (FOL)

outline: we will now consider

- the syntax
- the semantics
- the proof theory
- the meta-theory

of FOL

FOL Syntax

definition: a FOL-alphabet L consists of the following symbols

- variables $x_0, x_1, x_2, \ldots$
- constants $c_0, c_1, c_2, \ldots$
- function symbols $f_0, f_1, f_2, \ldots$ with arity $\alpha(f_i) \geq 1$
- predicate or relation symbols $P_0, P_1, P_2, \ldots$ with arity $\alpha(P_i) \geq 1$
- logical connectives $\neg, \wedge, \ldots$
- quantifier symbols $\exists, \forall$
- equality symbol $=$
- punctuation symbols $(,), \ldots$

FOL Syntax

definition: the set of FOL-terms on the alphabet L is inductively defined by

$$t ::= x \mid c \mid f(t_0, \ldots, t_n),$$

where x ranges over constants, x over variables and f over function symbols of arity n

examples:

- $+(square(\sin(x)), square(\cos(x)))$ stands for $\sin^2(x) + \cos^2(x)$
- $pop(push(41, empty))$ stands for the actions of pushing 41 to the empty stack and popping it again

FOL Syntax

definition: the set of FOL-formulae is inductively defined by

$$\phi ::= t_0 = t_1 \mid P(t_0, \ldots, t_{n-1}) \mid \neg\phi \mid \phi \wedge \psi \mid \exists x.\phi$$

remarks:

- $t_0 = t_1$ and $P(t_0, \ldots, t_{n-1})$ are atomic formulae
- $\forall x.\phi$ is syntactic sugar for $\neg\exists x.\neg\phi$

examples:

- the two translated examples are FOL-formulae
- $\sin^2(x) + \cos^2(x) = 1$ is an atomic formula
- $s = \mathit{empty} \vee \exists x.\exists t.s = \mathit{push}(x, t)$ is a formula

FOL Syntax

variables: they can occur bound or free in a formula.

example: in $(\forall x.P(x, y)) \vee Q(x)$ the blue variable occurs bound (by the quantifier) and the orange variables occur free

scope: the scope of a quantifier is the "region" where he binds variables

examples:

- in the above example, the scope of $\forall x$ is $\forall x.P(x, y)$
- in $\exists x.P(x) \wedge \exists x.\neg P(x)$ the two occurrences of x are in the scope of different quantifiers

analogy: in programs, parameter bindings are determined by the block structure

FOL Syntax

substitution: we write $\phi[t/x]$ to denote that each free occurrence of variable x in formula ϕ is replaced (or substituted) by term t.

remark: this can be defined inductively wrt the structure of terms and formulae

examples:

$$x[t/x] = t \qquad x[t/y] = x$$

$$f(x, g(a, x, y))[h(b)/x] = f(h(b), g(a, h(b), y))$$

$$(P(x) \vee Q(x, y)[a/x] = (P(a) \vee Q(a, y))$$

$$(\exists x.P(x, y))[a/y] = \exists x.P(x, a)$$

$$(\exists x.P(x, y))[f(x)/y] = \exists x.P(x, f(z))$$

FOL Semantics

observation: formulas of FOL can again be true or false; terms can't

question: what is the meaning of a term?

idea:

- interpret constants and variables over a domain (a set)
- interpret function symbols by mappings on the domain
- interpret predicate symbols by relations on the domain
- extend this definition to logical connectives
- understand universal quantification as infinite conjuction and existential quantification as infinite disjunction

FOL Semantics

definition: let L be a FOL-alphabet. An L-structure is a pair $A = (A, \alpha)$ where

- A is a (non-empty) set
- α associates every constant c, function symbol f, predicate symbol P with an element c^A, function f^A, relation P^A in A

definition: a valuation is a mapping $\beta : \{x_n : n \in \mathbb{N}\} \to A$. we define

$$\beta[a/x](y) = \begin{cases} \beta(y) & \text{if } y \neq x \\ a & \text{if } y = x \end{cases}$$

FOL Semantics

definition: an interpretation I is a pair (A, β) of a structure and a valuation. We inductively define $I(t)$ as

- $I(x) = \beta(x)$ for each variable x
- $I(c) = c^A$ for each constant c
- $I(f(t_0, \ldots, t_{n-1})) = f^A(I(t_0), \ldots, I(t_{n-1}))$ for each function symbol f of arity n and terms $t_0, \ldots, t_{n-1}$

definition: we set $I[a/x] = (A, \beta[a/x])$

FOL Semantics

definition: the satisfaction relation is inductively defined by

$$
\begin{array}{lll}
I \models t_0 = t_1 & \text{iff} & I(t_0) = I(t_1) \\
I \models P(t_1, \ldots, t_n) & \text{iff} & P^A(I(t_1), \ldots, I(t_n)) \\
I \models \neg\phi & \text{iff} & \text{not } I \models \phi \\
I \models \phi \wedge \psi & \text{iff} & I \models \phi \text{ and } I \models \psi \\
I \models \exists x.\phi & \text{iff} & I[a/x] \models \phi \text{ for some } a \in A \\
I \models \forall x.\phi & \text{iff} & I[a/x] \models \phi \text{ for all } a \in A
\end{array}
$$

definition: if $I \models \phi$ we say that I is a model of ϕ

FOL Proof Theory

FOL calculus: inference rules of PL plus

- introduction/elimination rule for universal quantification

$$\frac{\phi[a/x] \text{ true}}{\forall x.\phi \text{ true}}\forall_i \qquad \frac{\forall x.\phi \text{ true}}{\phi[t/x] \text{ true}}\forall_e$$

  where a is a fresh constant
- introduction/elimination rule for existential quantification

$$\frac{\phi[t/x] \text{ true}}{\exists x.\phi \text{ true}}\exists_i \qquad \frac{\exists x.\phi \text{ true} \qquad \begin{array}{c}\phi[a/x] \text{ true}\\ \vdots \\ \psi \text{ true}\end{array}}{\psi \text{ true}}\exists_e$$

  where a is a fresh constant

FOL Proof Theory

- inference rules for equality

$$\frac{}{t = t} =_i \qquad \frac{s = t \quad \phi[s/x]}{\phi[t/x]} =_e$$

Examples

example: $\forall x.\forall y.\phi \leftrightarrow \forall y.\forall x.\phi$ (one direction)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x.\forall y.\phi}\,1}{\forall y.\phi[a/x]}\,\forall_e}{\phi[b/y][a/x]}\,\forall_e}{\phi[a/x][b/y]}\,\text{subst}}{(\forall x.\phi)[b/y]}\,\forall_i}{\forall y.\forall x.\phi}\,\forall_i}{\forall x.\forall y.\phi \rightarrow \forall y.\forall x.\phi}\,\Rightarrow_i^1$$

Examples

example: $\exists x.(\phi \lor \psi) \leftrightarrow (\exists x.\phi \lor \exists x.\psi)$

$$\dfrac{\dfrac{}{\exists x.(\phi \lor \psi)}\,\text{ass} \qquad \dfrac{\dfrac{}{(\phi \lor \psi)[a/x]}\,1 \quad \dfrac{\dfrac{\dfrac{}{\phi[a/x]}\,2}{\exists x.\phi}\,\exists_i}{\exists x.\phi \lor \exists x.\psi}\,\lor_i \quad \dfrac{\dfrac{\dfrac{}{\psi[a/x]}\,3}{\exists x.\psi}\,\exists_i}{\exists x.\phi \lor \exists x.\psi}\,\lor_i}{\exists x.\phi \lor \exists x.\psi}\,\lor_e^{23}}{\exists x.\phi \lor \exists x.\psi}\,\exists_e^{1}$$

$$\dfrac{\dfrac{}{\exists x.\phi \lor \exists x.\psi}\,\text{ass} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\exists x.\phi}\,1 \quad \dfrac{}{\phi[a/x]}\,3}{\phi[a/x]}\,\exists_e^{3}}{(\phi \lor \psi)[a/x]}\,\lor_i}{\exists x.(\phi \lor \psi)}\,\exists_i \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\exists x.\psi}\,2 \quad \dfrac{}{\psi[b/x]}\,4}{\psi[b/x]}\,\exists_e^{4}}{(\phi \lor \psi)[b/x]}\,\lor_i}{\exists x.(\phi \lor \psi)}\,\exists_i}{\exists x.(\phi \lor \psi)}\,\lor_e^{12}$$

FOL Meta-Theory

theorem: our FOL-calculus is sound and complete with respect to the given semantics

but: satisfiability of FOL-formulae is no longer decidable

consequence: proofs in FOL require cleverness

however:

- there are proof search procedures for FOL that will prove a valid formula within finite time
- such procedures are highly relevant for software verification

A Catalogue of PL Validities

- commutativity

$$\phi \vee \psi \leftrightarrow \psi \vee \phi \qquad \phi \wedge \psi \leftrightarrow \psi \wedge \phi \qquad (\phi \leftrightarrow \psi) \leftrightarrow (\psi \leftrightarrow \phi)$$

- associativity

$$(\phi \vee \psi) \vee \chi \leftrightarrow \phi \vee (\psi \vee \chi) \qquad (\phi \wedge \psi) \wedge \chi \leftrightarrow \phi \wedge (\psi \wedge \chi)$$
$$((\phi \leftrightarrow \psi) \leftrightarrow \chi) \leftrightarrow (\phi \leftrightarrow (\psi \leftrightarrow \chi))$$

- distributivity

$$\phi \wedge (\psi \vee \chi) \leftrightarrow (\phi \wedge \psi) \vee (\phi \wedge \chi) \qquad \phi \vee (\psi \wedge \chi) \leftrightarrow (\phi \vee \psi) \wedge (\phi \vee \chi)$$
$$\phi \to (\psi \wedge \chi) \leftrightarrow (\phi \to \psi) \wedge (\phi \to \chi)$$

A Catalogue of PL Validities

- tertium non datur

$$\phi \vee \neg\phi$$

- idempotence

$$\phi \vee \phi \leftrightarrow \phi \leftrightarrow \phi \wedge \phi$$

- absorption

$$(\phi \wedge \psi) \vee \phi \leftrightarrow \phi \leftrightarrow (\phi \vee \psi) \wedge \phi$$

- de Morgan laws

$$\phi \wedge \psi \leftrightarrow \neg(\neg\phi \vee \neg\psi) \quad \phi \vee \psi \leftrightarrow \neg(\neg\phi \wedge \neg\psi)$$

- contraposition

$$(\phi \rightarrow \psi) \leftrightarrow (\neg\psi \rightarrow \neg\phi)$$

A Catalogue of PL Validities

- double negation

$$\phi \leftrightarrow \neg\neg\phi$$

  - transitivity

$$(\phi \to \psi) \land (\psi \to \chi) \to (\phi \to \chi)$$

  - isotonicity

$$(\phi \to \psi) \to (\phi \land \chi \to \psi \land \chi) \qquad (\phi \to \psi) \to (\phi \lor \chi \to \psi \lor \chi)$$

  - equivalence

$$(\phi \leftrightarrow \psi) \to (\phi \land \chi \leftrightarrow \psi \land \chi) \qquad (\phi \leftrightarrow \psi) \to (\phi \lor \chi \leftrightarrow \psi \lor \chi)$$

$$(\phi \leftrightarrow \psi) \to (\neg\phi \leftrightarrow \neg\psi)$$

A Catalogue of FOL Validities

- commutativity

$$\forall x.\forall y.\phi \leftrightarrow \forall y.\forall x.\phi \qquad \exists x.\exists y.\phi \leftrightarrow \exists y.\exists x.\phi$$

- associativity

$$\forall x.(\phi \wedge \psi) \leftrightarrow \forall x.\phi \wedge \forall x.\psi \qquad \exists x.(\phi \vee \psi) \leftrightarrow \exists x.\phi \vee \exists x.\psi$$

- de Morgan law

$$\forall x.\phi \leftrightarrow \neg\exists x.\neg\phi \qquad \exists x.\phi \leftrightarrow \neg\forall x.\neg\phi$$