The University of Sheffield

SCHOOL OF MATHEMATICS AND STATISTICS

Autumn Semester 2004–2005 2 hours

Discrete Foundations

*Answer **three** questions. You are advised **not** to answer more: if you do, only your best three answers will be counted.*

1 (i) Let $A = \{1, 3, 5, 6\}$, $B = \{2, 3, 6\}$ and $C = \{1, 4, 6\}$.
Find (a) $(A \cap B) \cup C$, (b) $(A \cup B) \setminus C$ and (c) $(A \setminus B) \cap C$. ***(9 marks)***

(ii) Draw four Venn diagrams showing three sets A, B and C in general position, and shade them to illustrate the following four sets: $A \setminus B$, $B \cup C$, $(A \setminus B) \setminus C$ and $A \setminus (B \cup C)$. ***(8 marks)***

(iii) (a) Let $A = \{a, b, c\}$, $B = \{p, q\}$. Find $A \times B$ (in the form $\{\ldots\}$, listing the elements). ***(2 marks)***

(b) The function $P : A \times B \to A$ is defined by $P(x, y) = x$ for all $x \in A$, $y \in B$. Give an example where P is not injective, and prove that P is always surjective. ***(4 marks)***

(iv) Let $\Sigma = \{a, b, c, \ldots, z\}$, and for each of the following functions $f, g, h : \Sigma^* \to \Sigma^*$ say whether the function is (a) injective and (b) surjective. Justify your answers. We define f, g, h by defining, for $a_1, \ldots, a_n \in \Sigma$:

$$
\begin{aligned}
f(a_1a_2\ldots a_{n-1}a_n) &= a_na_{n-1}\ldots a_2a_1; \\
g(a_1a_2\ldots a_{n-1}a_n) &= \begin{cases} a_1a_2\ldots a_n, & \text{if } n \text{ is even,} \\ a_1a_2\ldots a_{n-1}, & \text{if } n \text{ is odd;} \end{cases} \\
h(a_1a_2\ldots a_{n-1}a_n) &= \begin{cases} a_1a_2\ldots a_{n-1}, & \text{if } n > 1, \\ \varepsilon, & \text{if } n = 1. \end{cases}
\end{aligned}
$$

We define $f(\varepsilon) = g(\varepsilon) = h(\varepsilon) = \varepsilon$. ***(11 marks)***

2 (i) Prove by induction that

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4} \qquad (n = 1, 2, 3, \ldots).$$ ***(11 marks)***

(ii) We define a sequence (a_n) recursively by:

$$\begin{aligned} a_1 &= 1; \\ a_{n+1} &= a_n + \frac{1}{n(n+1)} \qquad (n \geqslant 1). \end{aligned}$$

(a) Calculate a_1, a_2, a_3, a_4 recursively (you may leave the answers as fractions). ***(3 marks)***

(b) Prove by induction that

$$a_n = \frac{2n-1}{n}, \qquad \text{for all positive integers } n.$$ ***(10 marks)***

(iii) Use truth tables to prove that one of the following is a tautology and the other one is a contradiction:

(a) $((P \vee Q) \Rightarrow (P \wedge Q)) \Rightarrow (P \Rightarrow Q)$. ***(5 marks)***

(b) $Q \wedge ((P \Rightarrow Q) \Rightarrow \neg Q)$. ***(5 marks)***

3 (i) Give the formal definition of the notion of a **finite automaton** $M = (Q, \Sigma, \delta, q_0, F)$, explaining what the five elements of this quintuple are, how the machine operates and what is meant by the **language** $L(M)$ **accepted by** M. ***(8 marks)***

(ii) Let $M = (Q, \Sigma, \delta, q_0, F)$ be a finite automaton in which $Q = \{q_0, q_1, q_2\}$, $\Sigma = \{0, 1, 2\}$, $F = \{q_2\}$ and δ is given by the following transition table.

δ	0	1	2
q_0	q_0	q_0	q_1
q_1	q_0	q_2	q_2
q_2	q_2	q_2	q_2

Draw the state diagram for M and describe the language $L(M)$. ***(8 marks)***

(iii) Design a finite automaton with input alphabet $\{0, 1\}$ which accepts precisely those strings which contain an odd number of 0s and an odd number of 1s. (For example, 01101011 is accepted, 10101 is not.) Specify your machine by drawing its state diagram. ***(8 marks)***

(iv) Give regular expressions corresponding to each of the following languages over the alphabet $\{a, b, c\}$:

(a) the set of all strings containing precisely two occurrences of the character b; ***(4 marks)***

(b) the set of all strings not containing the substring cab. ***(6 marks)***

4 (i) Give the formal definition of the notion of a **Mealy machine** and explain how such a machine operates. ***(8 marks)***

(ii) Design a Mealy machine with input alphabet $\Sigma = \{0, 1\}$ and the output alphabet $\Gamma = \{0, 1, 2\}$ such that the output digit at any stage is the sum of the last two input digits. (For example, the input 11001011 should produce the output 12101112.) Specify your machine by giving **both** a transition/output table **and** a state diagram. ***(8 marks)***

(iii) Minimize the Mealy machine with states p, q, r, s, t, u, v, initial state p and the following transition/output table.

		p	q	r	s	t	u	v
δ	0	t	u	s	s	q	q	t
δ	1	q	p	s	r	p	v	u
ω	0	0	0	1	0	1	0	0
ω	1	1	1	0	1	0	1	1

(10 marks)

(iv) Consider the grammar $G = (V, N, \Sigma, P, S)$ with $N = \{S, X\}$, $\Sigma = \{a, b, c\}$, starting symbol S and the production rules

$$S \to XaSbX, \qquad S \to c, \qquad X \to a, \qquad X \to b.$$

Is this a context-free grammar? Is this a regular grammar? Find the language generated by this grammar. ***(8 marks)***

End of Question Paper