

Discrete Foundations — Question Sheet 3

Please complete this by Friday 27 October, when the solutions will be given. The starred questions are for COM6853 students.

1. Recall that a relation R on a set A is said to be:

- reflexive if xRx for every $x \in A$;
- symmetric if xRy implies yRx for every $x, y \in A$;
- transitive if xRy and yRz imply xRz for every $x, y, z \in A$.

Let A be the power set of \mathbb{R} , i.e. the set of all sets of real numbers, and consider the relation \subseteq on A . Is this relation (a) reflexive, (b) symmetric, (c) transitive?

2. Prove directly that

$$\frac{200}{700^2} < \frac{1}{500} - \frac{1}{700} < \frac{200}{500^2}$$

3. Prove that

$$\frac{x(x-1)^2 - 5x + 8}{x+2} = (x-2)^2,$$

when $x \neq -2$. What happens at $x = -2$?

4. Is the following proof correct ? If not, why not?

Theorem $1 = 2$.

Proof

$$\begin{aligned}x = 2 &\Rightarrow x - 1 = 1 \\&\Rightarrow (x - 1)^2 = 1 \\&\Rightarrow (x - 1)^2 = x - 1 \\&\Rightarrow x^2 - 2x + 1 = x - 1 \\&\Rightarrow x^2 - 2x = x - 2 \\&\Rightarrow x(x - 2) = x - 2 \\&\Rightarrow \frac{x(x - 2)}{x - 2} = \frac{(x - 2)}{x - 2} \\&\Rightarrow x = 1\end{aligned}$$

5. Prove by induction that

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}.$$

* How could you use this result to calculate $\sum_{k=1}^n k^3$? Hint: consider

$$\sum_{k=1}^n (k+1)^4 - \sum_{k=1}^n k^4.$$

6. * Prove by induction that $n^3 < 3^n$ for $n \geq 4$.

7. Prove by induction that $11^n - 4^n$ is divisible by 7 for all $n \in \mathbb{N}$.
8. Prove without differentiation that $x + \frac{1}{x} \geq 2$ when $x > 0$. What happens at $x = 0$?
9. Prove by exhaustion that $x^2 + y^2 = 11^2$ has no integer solution with $x, y > 0$.
10. Prove by contradiction that $\sqrt{2}$ is irrational — i.e. that $\sqrt{2}$ cannot be expressed in the form p/q for positive integers p, q . You may use without proof the fact that, for an integer a , if a^2 is even, then a is even.