COM3502/4502/6502
SPEECH PROCESSING

Lecture 15
The Fourier Transform

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Phase in Spectral Analysis

- In the last lecture it was shown how the spectrum could be computed by 'cosine correlation' (*Pd Example 14-6*)
- However, this only worked because the target signals all had 'zero phase'
- The correlation between two cosines varies according to the phase difference between them
  - in-phase → maximum correlation
  - 90° phase difference → zero correlation
  - 180° phase difference → maximum negative correlation
- In fact the correlation between two cosines varies as a cosine!

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Phase in Spectral Analysis

Multiplication of two cosines → $\cos(a).\cos(b) = \frac{1}{2}\left(\cos(a-b) + \cos(a+b)\right)$

Multiplication of two cosine waves → $\cos(\omega t).\cos(\omega t) = \frac{1}{2}\left(\cos(0) + \cos(2\omega t)\right)$

$= \frac{1}{2}\left(1 + \cos(2\omega t)\right)$ ← Constant + cosine at twice the frequency

Multiplication of two cosine waves differing in phase → $\cos(\omega t).\cos(\omega t - \phi) = \frac{1}{2}\left(\cos(\phi) + \cos(2\omega t - \phi)\right)$

Cosine of the phase difference + cosine at twice the frequency

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Phase in Spectral Analysis

Correlation of two cosine waves differing in phase →

$$q = \sum_{i=0}^{N-1} \cos(\omega nT).A\cos(\omega nT - \phi)$$

$$= \sum_{i=0}^{N-1} \left(\frac{A}{2}\left(\cos(\phi) + \cos(2\omega t - \phi)\right) \right)$$

$$= \sum_{i=0}^{N-1} \left(\frac{A}{2}\left(\cos(\phi)\right) \right) + \sum_{i=0}^{N-1} \left(\frac{A}{2}\left(\cos(2\omega t - \phi)\right) \right)$$

$$= \sum_{i=0}^{N-1} \left(\frac{A}{2}\left(\cos(\phi)\right) \right)$$

$$= \frac{A.N}{2}\cos(\phi)$$

$$= \alpha\cos(\phi)$$

Cosine of the phase difference

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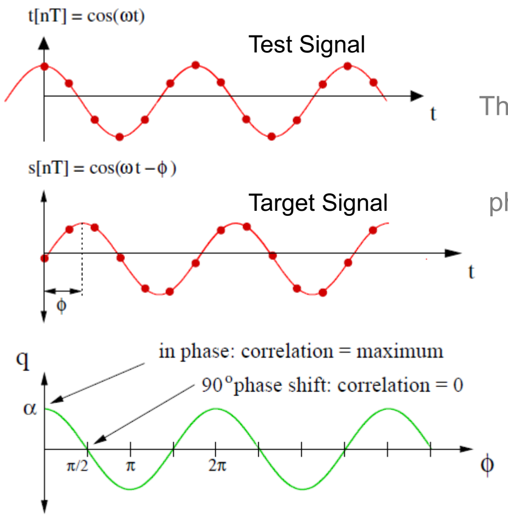
Phase in Spectral Analysis



This means that cosine correlation cannot detect a $\pi/2$ ($90°$) phase shift in a target signal!

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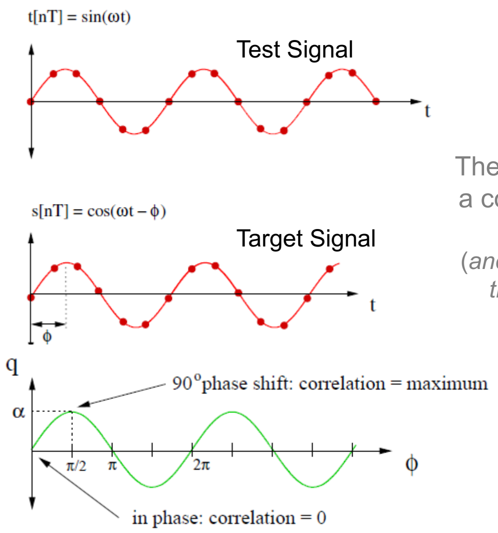
Sine Correlation



The correlation between a cosine and a sine is a sine!

(and it varies according to the phase difference between them)

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Sine Correlation

Correlation of sine wave with cosine wave

$$q=\sum_{i=0}^{N-1}\sin(\omega nT).A\cos(\omega nT-\phi)$$

$$=\sum_{i=0}^{N-1}\cos(\omega nT-\pi/2).A\cos(\omega nT-\phi)$$

$$=\sum_{i=0}^{N-1}\left(\frac{A}{2}(\cos(\phi-\pi/2)+\cos(2\omega t-\phi-\pi/2))\right)$$

$$=\sum_{i=0}^{N-1}\left(\frac{A}{2}(\sin(\phi)+\sin(2\omega t-\phi))\right)$$

$$=\sum_{i=0}^{N-1}\left(\frac{A}{2}(\sin(\phi))\right)+\sum_{i=0}^{N-1}\left(\frac{A}{2}(\sin(2\omega t-\phi))\right)$$

$$=\sum_{i=0}^{N-1}\left(\frac{A}{2}(\sin(\phi))\right)$$

$$=\alpha\sin(\phi)$$

Sine of the phase lag

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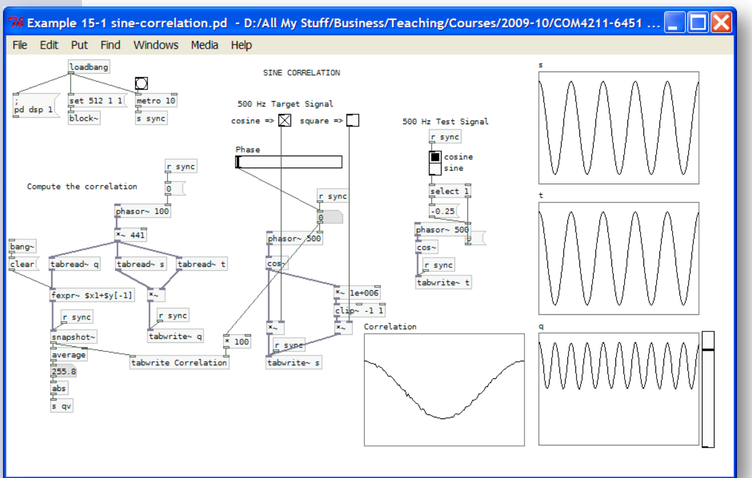
Sine Correlation



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Sine *and* Cosine Correlation

- Cosine correlation with cosine is a cosine
- Sine correlation with cosine is a sine
- Sine and cosine correlations are 90° ($\pi/2$) out of phase
- So can cosine and sine correlation be combined in some useful way?

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Amplitude and Phase

- Correlate a signal with *both* sines *and* cosines …

$$cosinecorrelation = \alpha.\cos(\phi)$$
$$sinecorrelation = \alpha.\sin(\phi)$$

- Note …

$$(\alpha.\cos(\phi))^2 + (\alpha.\sin(\phi))^2 = \alpha^2$$

- Hence the amplitude of the sinusoidal component independent of phase is given by …

$$\alpha = \sqrt{(cosinecorrelation)^2 + (sinecorrelation)^2}$$

- The phase of this component is given by …

$$\tan(\phi) = \alpha.\sin(\phi)/\alpha.\cos(\phi)$$

$$\therefore \phi = \tan^{-1}(sinecorrelation/cosinecorrelation)$$

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General Spectral Analysis Algorithm

- Recall that any periodic signal can be expressed as the sum of a fundamental sinusoid and its harmonics (*Fourier*)
- The individual components at a frequency $\Omega = p\omega$ can be found by correlating $s(nT)$ with $\cos(\Omega nT)$ and $\sin(\Omega nT)$
- Let $c(\Omega)$ be the cosine correlation and $s(\Omega)$ the sine correlation …

$$c(\Omega) = \sum_{n=0}^{N-1} s(nT).\cos\left(\frac{2\pi np}{N}\right) \quad p = 0, 1, \ldots, N\text{-}1$$

$$s(\Omega) = \sum_{n=0}^{N-1} s(nT).\sin\left(\frac{2\pi np}{N}\right) \quad p = 0, 1, \ldots, N\text{-}1$$

$$\mathrm{a_p} = \sqrt{c(\Omega)^2 + s(\Omega)^2}$$

$$\phi_p = \tan^{-1}\left(\frac{s(\Omega)}{c(\Omega)}\right)$$

- This is the '**Discrete Fourier Transform**' (*DFT*)

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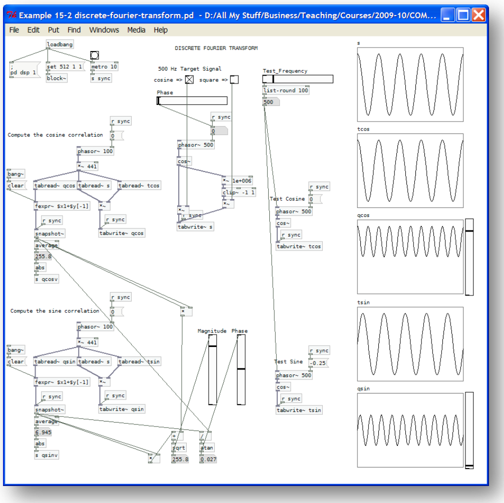
Discrete Fourier Transform



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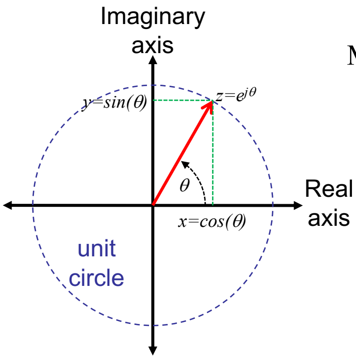
Complex Numbers: *A Reminder*

$$z = x + jy$$



$$\text{Magnitude } |z| = \sqrt{x^2 + y^2}$$

$$\text{Phase } \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

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Complex Formulation of the DFT

- The DFT is often expressed using 'complex number notation'
- The cosine and sine correlations are associated with the real and imaginary parts of a complex number …

$$S_p = \sum_{n=0}^{N-1} s(nT).\cos\left(\frac{2\pi np}{N}\right) - j\sum_{n=0}^{N-1} s(nT).\sin\left(\frac{2\pi np}{N}\right)$$

$$= \sum_{n=0}^{N-1} s(nT)\left[\cos\left(\frac{2\pi np}{N}\right) - j\sin\left(\frac{2\pi np}{N}\right)\right]$$

$$= \sum_{n=0}^{N-1} s(nT).e^{-j\left(\frac{2\pi np}{N}\right)}$$

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Complex Formulation of the DFT

- Hence the DFT can be expressed as …

$$S_p = \sum_{n=0}^{N-1} s(nT).e^{-j\left(\frac{2\pi np}{N}\right)} \qquad p = 0, 1, \ldots, N-1$$

… where S_p is a complex number whose magnitude and phase correspond to that of the spectrum of $s(nT)$ at a frequency p/NT

- Note also the 'inverse DFT' …

$$s(nT) = \frac{1}{N} \sum_{p=0}^{N-1} S_p.e^{j\left(\frac{2\pi np}{N}\right)} \qquad n = 0, 1, \ldots, N-1$$

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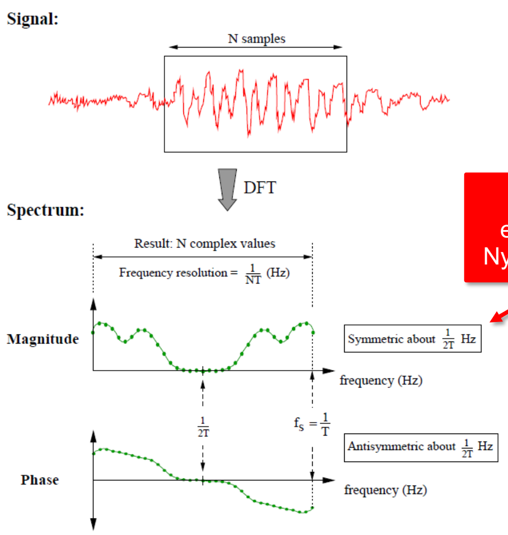
Using the DFT



This is the explanation for Nyquist's theorem

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Time *vs.* Frequency Resolution

Frequency resolution = $1/NT$ *(Hz)*

- Increasing the analysis frame N
  - decreases the spacing between the spectral components
  - reduces the ability to respond to changes in the signal
- Hence *large* N leads to 'narrowband analysis'
  - good spectral resolution
  - poor time resolution

- Decreasing the analysis frame N
  - increases the spacing between the spectral components
  - increases the ability to respond to changes in the signal
- Hence *small* N leads to 'wideband analysis'
  - good time resolution
  - poor spectral resolution

This is the *time-frequency trade-off* we saw in Lecture 4

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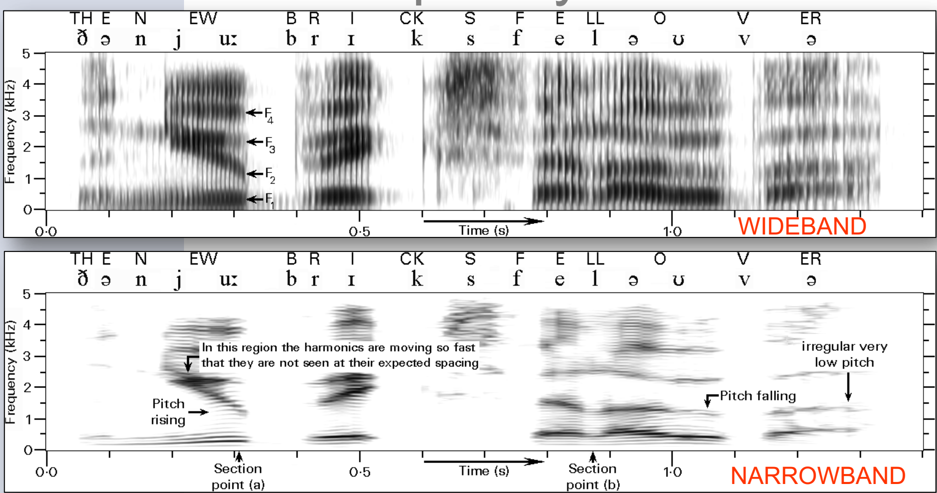
Time *vs.* Frequency Resolution



Taken from: Holmes, J. N., & Holmes, W. (2002). *Speech Synthesis and Recognition*: Taylor & Francis.

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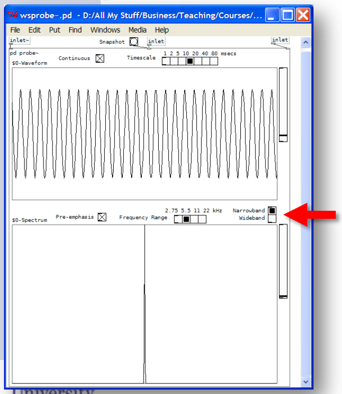
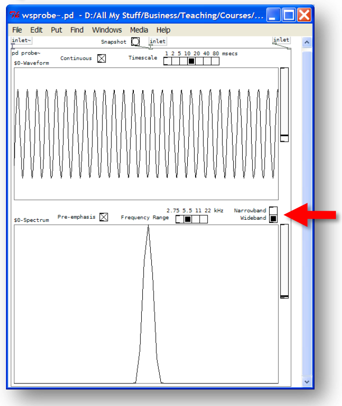
Time *vs.* Frequency Resolution

In [wsprobe~] …

- narrowband analysis uses a frame/block size of 8192 samples (*~190 msecs*)
- wideband analysis uses a frame/block size of 512 samples (*~12 msecs*)





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Implicit Periodicity in the DFT

- The DFT computes the spectrum at N evenly spaced discrete frequencies
- The algorithm assumes periodicity outside the analysis frame (*with a period equal to the frame length*)
- This means that discontinuities will arise for …
  - periodic signals with a *non-integer* number of cycles in the analysis frame
  - all aperiodic signals
  - all stochastic signals
- Such discontinuities give rise to unwanted spectral components

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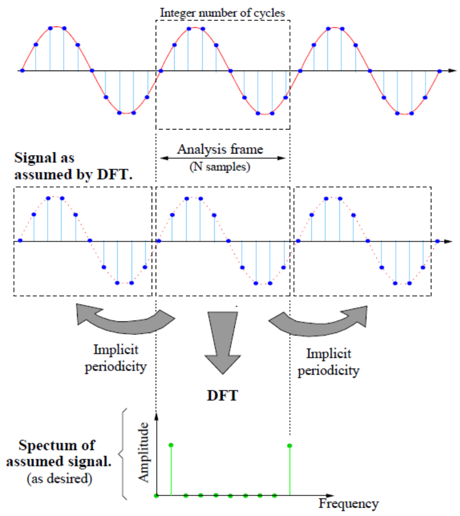
Implicit Periodicity in the DFT



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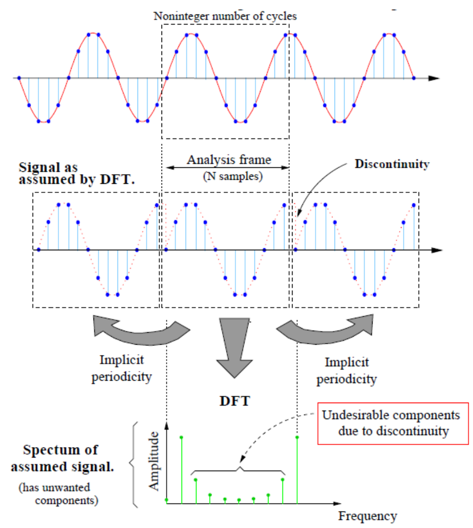
Implicit Periodicity in the DFT



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Windowing

- The discontinuities arising from segmenting the signal into frames distorts the spectrum
- The distortion can be reduced by multiplying each signal frame with a 'window function'
- The most common window function is the 'Hamming Window' (*proposed by Richard W. Hamming*) …

$$w(nT) = 0.54 - 0.46.\cos\left(\frac{2\pi n}{N-1}\right)$$

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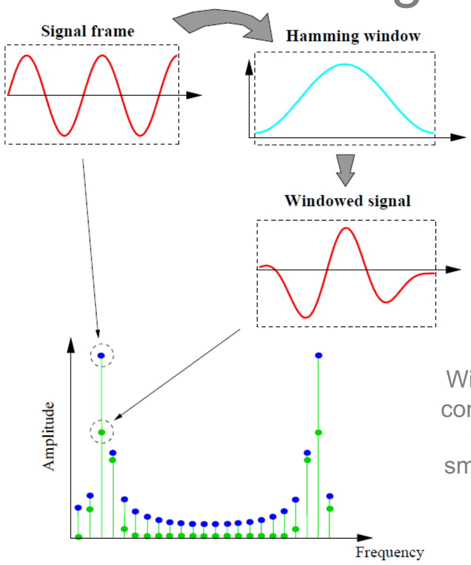
Windowing



Windowing attenuates the components caused by the discontinuity, but also smears the spectral peaks

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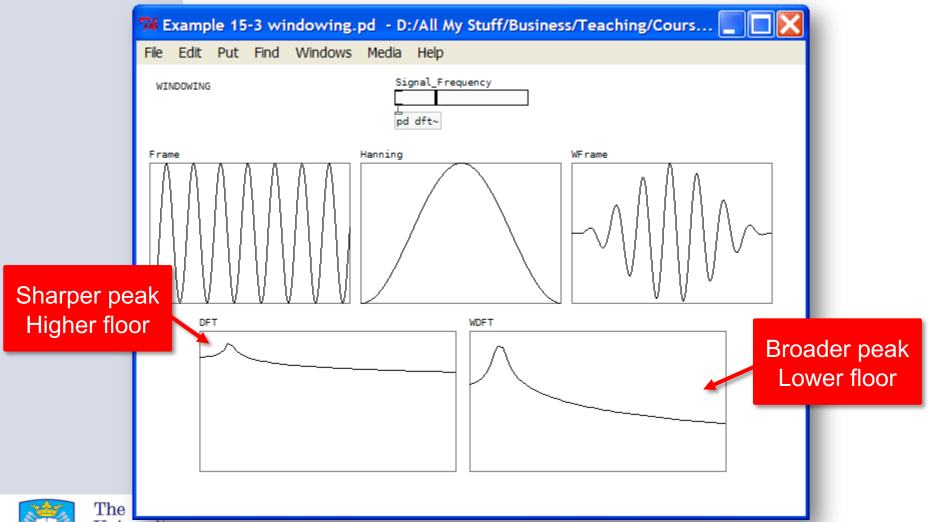
Windowing



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The Fast Fourier Transform (FFT)

- Implementation of the DFT requires the order of N^2 multiply-add operations
- By exploiting symmetry, it is possible to devise an algorithm that requires only $N\log_2 N$ multiply-add operations
  - e.g. for $N=2048$, the result is ~100x faster
- This more efficient algorithm is the … '**Fast Fourier Transform**' (*FFT*)
- The FFT requires that the window/frame should be a power of 2 in size
- This can be achieved by …
  - choosing the appropriate analysis frame size, and/or
  - zero-padding a frame to the nearest power of 2

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The FFT in Pure Data

- There are two key FFT objects available in Pd …
  - [rfft~] computes the forward transform
  - [ifft~] computes the inverse transform
- We saw both [rfft~] and [ifft~] in action in Example 12-3
- The magnitude spectrum can be computed by squaring the real and imaginary outputs, then taking the square root [sqrt~]
- Alternatively, the Pd object [framp~] outputs frequencies and amplitudes directly
- [rfft~] followed by [framp~] is used in [wsprobe~]

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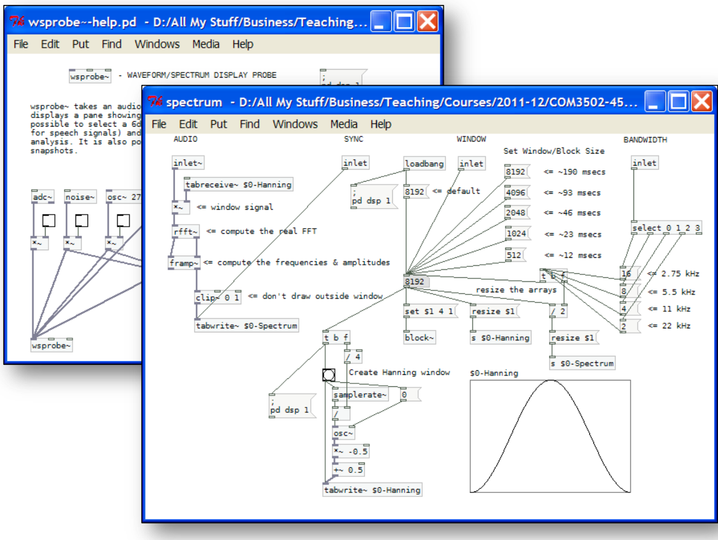
The FFT in Pure Data



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This lecture has covered …

- Phase in spectral analysis
- Sine and cosine correlation
- The discrete Fourier transform (*DFT*)
- Time versus frequency resolution
- Windowing
- The fast Fourier transform (*FFT*)

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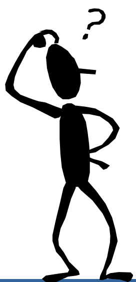
Any Questions ?



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Next time …

The 'Z' Transform