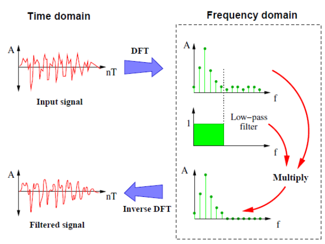
COM3502/4502/6502
SPEECH PROCESSING

Lecture 16
The 'Z' Transform

Filtering

- Recall that a signal may be filtered by …
  1. transforming it into the frequency domain (*using DFT*)
  2. multiplying the signal spectrum by the filter spectrum
  3. transforming it back into the time domain (*using IDFT*)



- An alternative is to characterise the action of a filter in the time domain (*i.e. on the waveform itself*)
- This is done using 'difference equations'

Difference Equations

- Consider a linear filter where …
  - *x[nT]* is the input
  - *y[nT]* is the output (*response*)

- '**Difference equations**' relate the current filter output to the current and past inputs *and* outputs

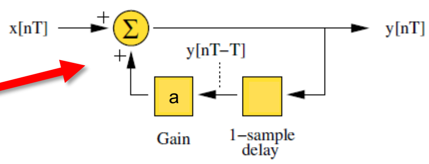
3

Difference Equations: *an example*

Difference equation

$$y[nT] = ay[(n-1)T] + x[nT]$$

Equivalent processing diagram



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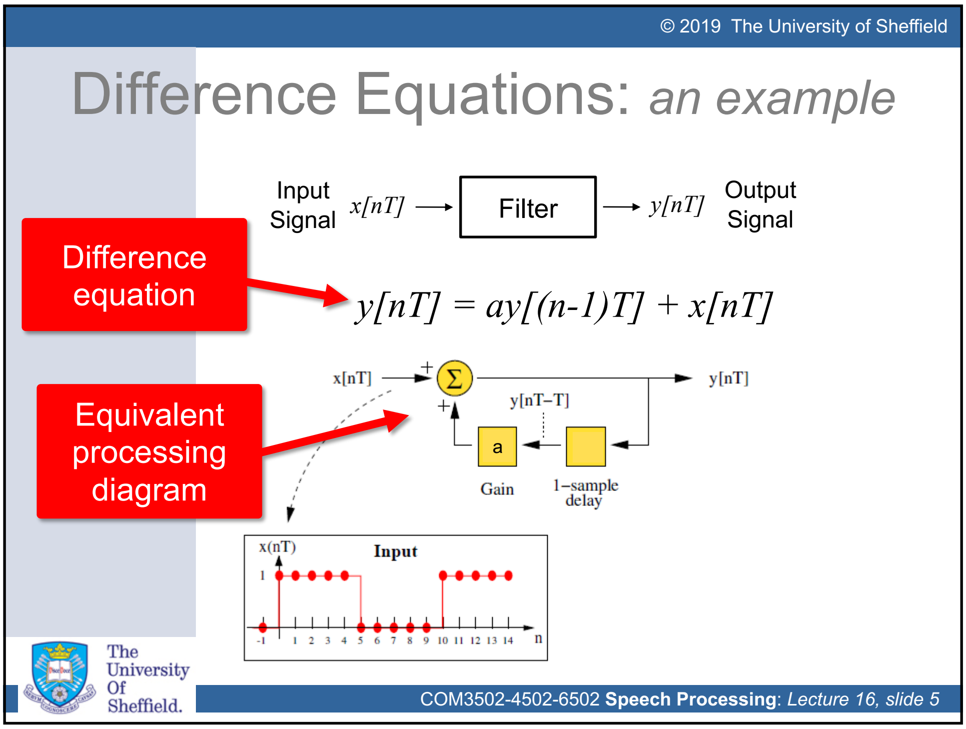
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Difference Equations: an example
Input Signal
x[nT]
Filter
y[nT]
Output Signal
Difference equation
y[nT] = ay[(n-1)T] + x[nT]
Equivalent processing diagram
x[nT]
y[nT]
y[nT–T]
a
Gain
1–sample delay
x(nT)
Input
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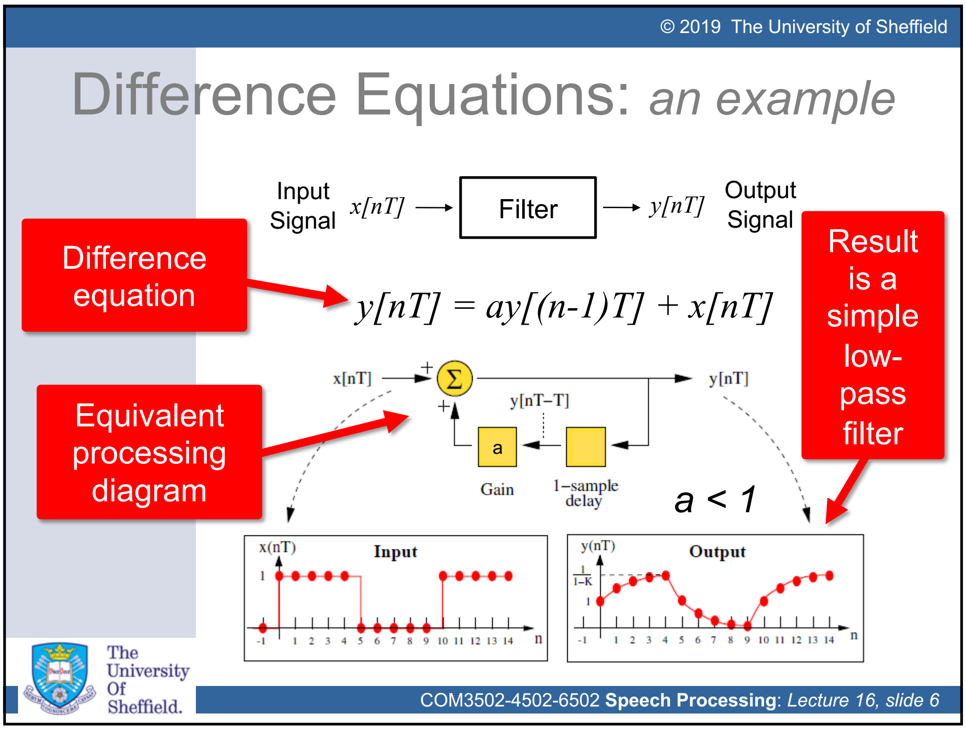
5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Difference Equations: an example
Input Signal
x[nT]
Filter
y[nT]
Output Signal
Difference equation
y[nT] = ay[(n-1)T] + x[nT]
Equivalent processing diagram
x[nT]
y[nT]
y[nT–T]
a
Gain
1–sample delay
Result is a simple low-pass filter
a < 1
x(nT)
Input
y(nT)
Output
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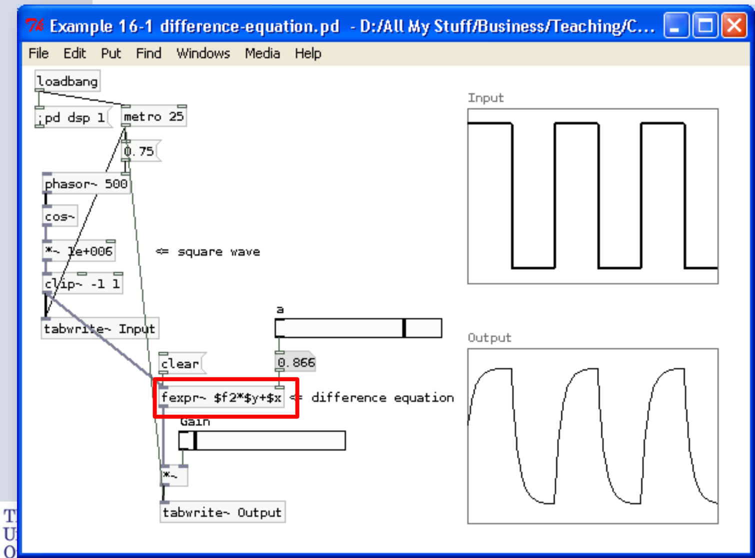
6

Difference Equations: *an example*



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7

Linear Time-Invariant Filters

- A 'linear filter' obeys the principle of super-position …
  - if $x_1[kT]$ yields $y_1[kT]$
  - and input is $ax_1[kT] + bx_2[kT]$
  - then $y[kT] = ay_1[kT] + by_2[kT]$
- The response of a 'time-invariant filter' is the same for all times …
  - if $x[kT]$ yields $y[kT]$
  - then $x[(k\text{-}k_0)T]$ yields $y[(k\text{-}k_0)T]$

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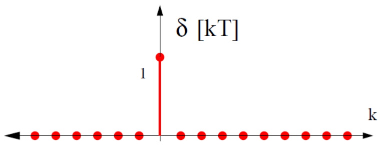
8

Linear Time-Invariant Filters

- Linearity and time-invariance allows the input (*and the response*) of a filter to be constructed from a weighted sum of time-shifted impulses $\delta[kT]$
  where $\delta[kT]$ is 1 when $k = 0$ (and 0 otherwise)



- Time-invariance and super-position thus allows the input signal to be expressed as …

$$x[kT] = \sum_{n=-\infty}^{\infty} x[nT].\delta[(k-n)T]$$

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Linear Time-Invariant Filters

- Given the '**impulse response**' of a filter $h[kT]$, the output is …

$$y[kT] = \sum_{n=-\infty}^{\infty} x[nT].h[(k-n)T]$$

- This is the '**discrete convolution**' of the input signal with the filter impulse response, and it can be written as …

$$y[kT] = x[kT] * h[kT]$$

convolution operator

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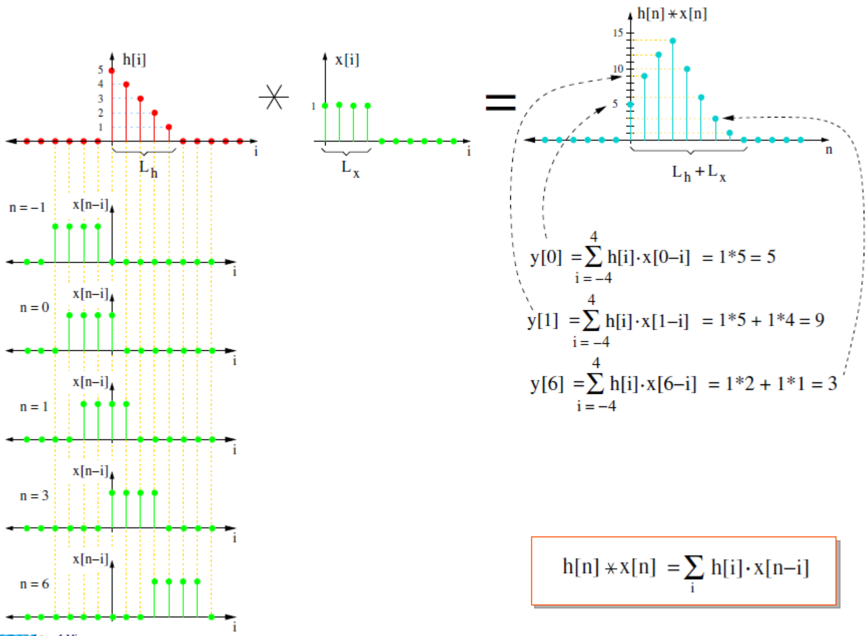

$$y[0] = \sum_{i=-4}^{4} h[i] \cdot x[0-i] = 1*5 = 5$$

$$y[1] = \sum_{i=-4}^{4} h[i] \cdot x[1-i] = 1*5 + 1*4 = 9$$

$$y[6] = \sum_{i=-4}^{4} h[i] \cdot x[6-i] = 1*2 + 1*1 = 3$$

$$h[n] * x[n] = \sum_{i} h[i] \cdot x[n-i]$$

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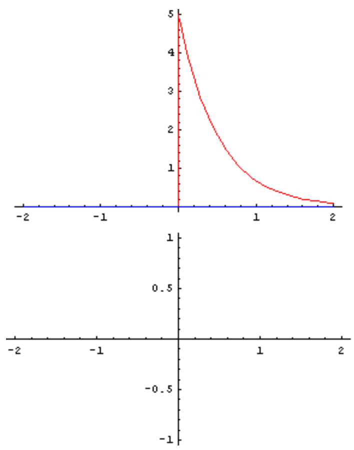
11

Convolution



http://en.wikipedia.org/wiki/Convolution

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The Z-Transform

- Analysis of filters in the time domain (*i.e. using difference equations*) is cumbersome
- The 'Z-transform' is a power series representation of a discrete-time sequence
- E.g. for the sequence $x[0]\ x[1]\ x[2]\ x[3]$, the Z transform simply multiplies each coefficient in the sequence by a power of z corresponding to its index …

$$X(z) = x[0].z^{0} + x[1].z^{-1} + x[2].z^{-2} + x[3].z^{-3}$$

- By convention, *negative* powers of z are used for *positive* time indices (*i.e.* z^{-n} *represents a delay of* n *samples*)

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13

The Z-Transform

- For a general sequence, the Z transform is written as …

$$X(z) = \sum_{k=-\infty}^{\infty} x[kT].z^{-k}$$

- The transform is denoted by the $Z\{.\}$ operator …

$$X(z) = Z\{x[kT]\}$$

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14

The Z-Transform

Key properties of the transform are …

– linearity

$$Z\{ax_1[k]+bx_2[k]\}=aZ\{x_1[k]\}+bZ\{x_2[k]\}$$

– time-shifting

$$Z\{x[k-k_0]\}=z^{-k_0}Z\{x[k]\}$$

– convolution

$$Z\{x[k]*y[k]\}=Z\{x[k]\}.Z\{y[k]\}$$

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The Z-Transform

- As we saw, the output of a linear filter is described by the discrete convolution …

$$y[k]=x[k]*h[k]$$

- This can be expressed using the Z transform as a product …

$$Y[z]=X[z].H[z]$$

- Hence the 'z-plane transfer function' $H[z]$ may be written as …

$$H[z]=\frac{Y[z]}{X[z]}$$

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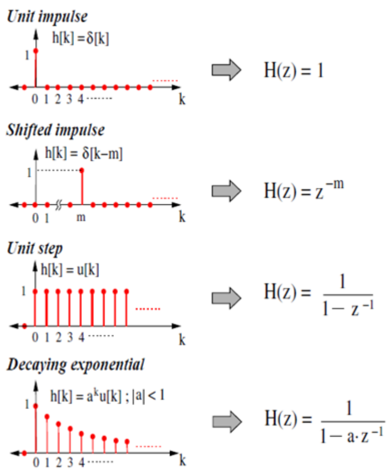
Common Z-Transform Pairs



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17

Link with the Fourier Transform

Z-Transform

$$Z\{x[kT]\} = \sum_{k=-\infty}^{\infty} x[kT].z^{-k}$$

- If $z>1$, then the transform is equivalent to multiplying the sequence by a *rising* exponential
- If $0<z<1$, then the transform is equivalent to multiplying the sequence by a *falling* exponential
- If z is a complex number *lying on the unit circle*, then the transform is equivalent to multiplying the sequence by a real cosine and imaginary sinusoid

  i.e. this is directly equivalent to the Fourier transform!
- Hence, given a filter transfer function $H[z]$, the filter frequency response can be evaluated at any frequency ω by setting …

$$z = e^{j\omega T}$$

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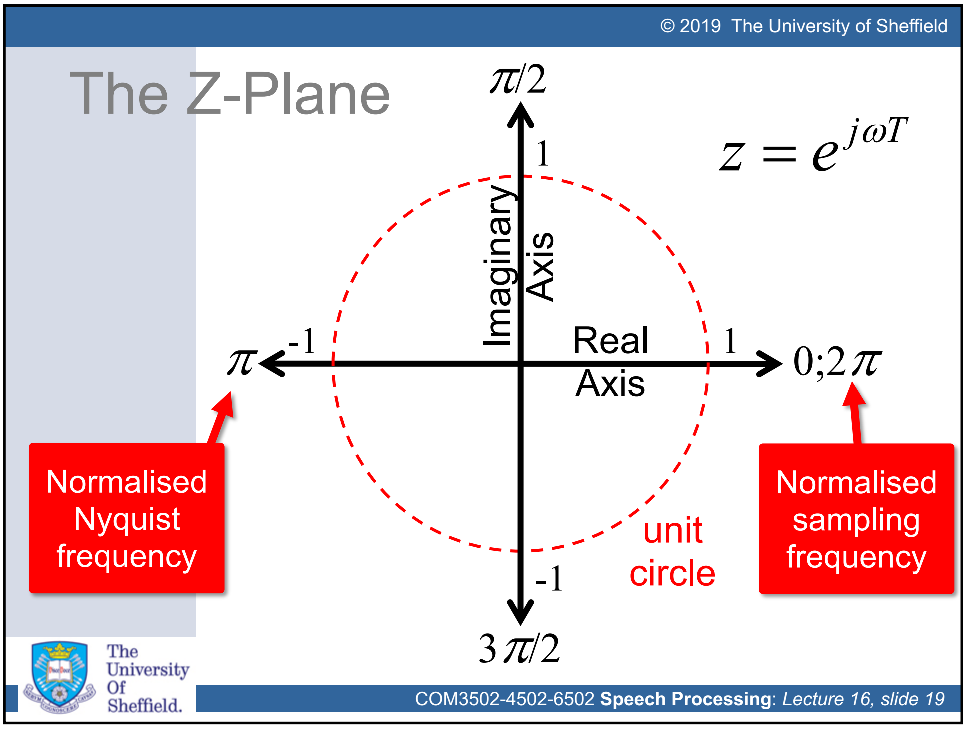
18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The Z-Plane
π/2
1
$z = e^{j\omega T}$
Imaginary Axis
Real Axis
π
-1
1
0;2π
Normalised Nyquist frequency
Normalised sampling frequency
unit circle
-1
3π/2
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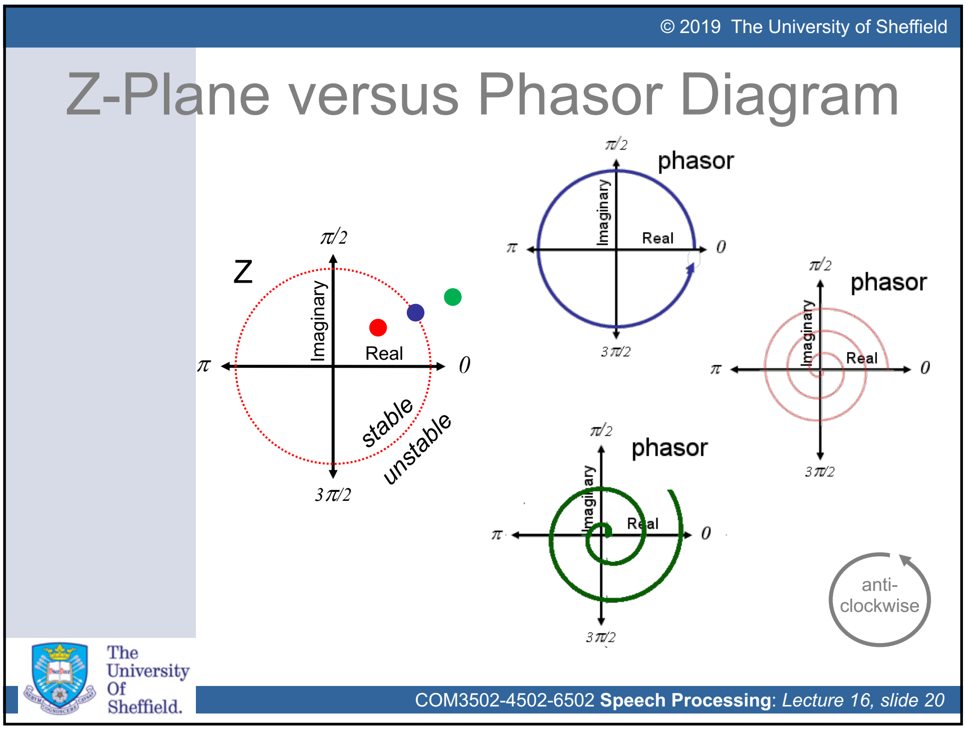
19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Z-Plane versus Phasor Diagram
Z
π/2
Imaginary
Real
π
0
3π/2
stable
unstable
phasor
phasor
phasor
anti-clockwise
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20

Poles and Zeros

- Linear filter transfer functions (*arising from general difference equations*) can be written as the ratio of two polynomials in z …

$$H[z] = \frac{P[z]}{Q[z]}$$

- The values of z for which $P[z]=0$ are known as the '**zeros**' of $H[z]$
- The values of z for which $Q[z]=0$ are known as the '**poles**' of $H[z]$
- Poles correspond to frequencies at which a filter transfer function tends to infinity (*i.e.* '*resonances*')
- Zeros correspond to frequencies at which a filter transfer function tends to zero (*i.e.* '*anti-resonances*')
- A filter is completely characterised by its poles and zeros

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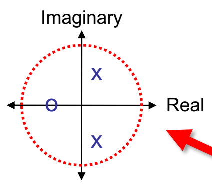
Poles and Zeros

- The z-domain polynomials are often written in terms of their poles and zeros z_i …

Zeros → $$P[z] = \prod_{i=1}^{p} (z - z_i) \qquad Q[z] = \prod_{i=1}^{q} (z - z_i)$$ ← Poles

- The poles and zeros correspond to the '**roots**' of the difference equations
- They can be visualised by plotting their positions in the '**Z-Plane**' …

Zeros are signified by 'o's



Poles are signified by 'x's

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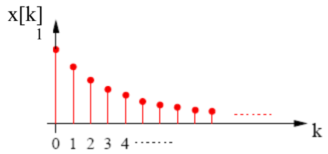
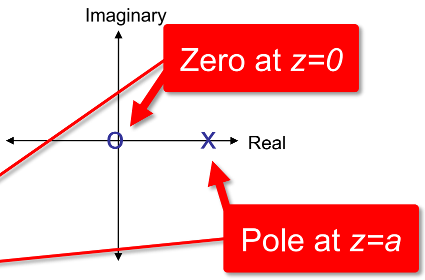
22

Poles and Zeros: *an example*

$$x[k] = 0 \text{ for } k < 0$$
$$x[k] = a^k \text{ otherwise}$$



$$X(z) = \sum_{k=-\infty}^{\infty} x[k] z^{-k}$$
$$= \sum_{k=0}^{\infty} a^k z^{-k}$$
$$= \sum_{k=0}^{\infty} (a.z^{-1})^k$$
$$= \frac{1}{1 - a.z^{-1}}$$
$$= \frac{z}{z-a}$$



Poles and Zeros

- The magnitude response of a filter can be quickly understood based on the location of its poles and zeros
- By starting with a pole/zero plot, it is possible to design a filter and obtain its transfer function very easily …

$$H[z] = \frac{\prod_{i=1}^{p} (z - z_i)}{\prod_{i=1}^{q} (z - z_i)}$$

$$= \frac{\prod \text{"distance of zeros from unit circle"}}{\prod \text{"distance of poles from unit circle"}}$$

Filter Frequency Response

- The frequency response of a filter can be plotted using the distances from the 'unit circle' to the poles and zeros
- While moving around the unit circle …
  - if close to a zero, then the magnitude is small
  - if close to a pole, then the magnitude is large
- If a *zero* is on the unit circle, then the frequency response is zero at that point
- If a *pole* is on the unit circle, then the frequency response goes to infinity at that point

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Filter Response: *an example*

- Consider the simple averaging filter with the following difference equation …

$$y[n] = ay[n-1] + x[n]$$

- Taking Z transforms, this becomes …

$$Y = aYz^{-1} + X$$

- Hence the transfer function (Y/X) is given by …

$$Y - aYz^{-1} = X$$

$$Y\left(1 - az^{-1}\right) = X$$

$$\frac{Y}{X} = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

Zero at $z=0$

Pole at $z=a$

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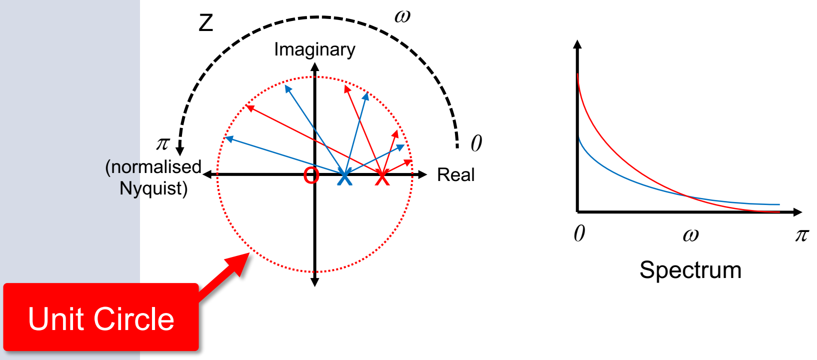
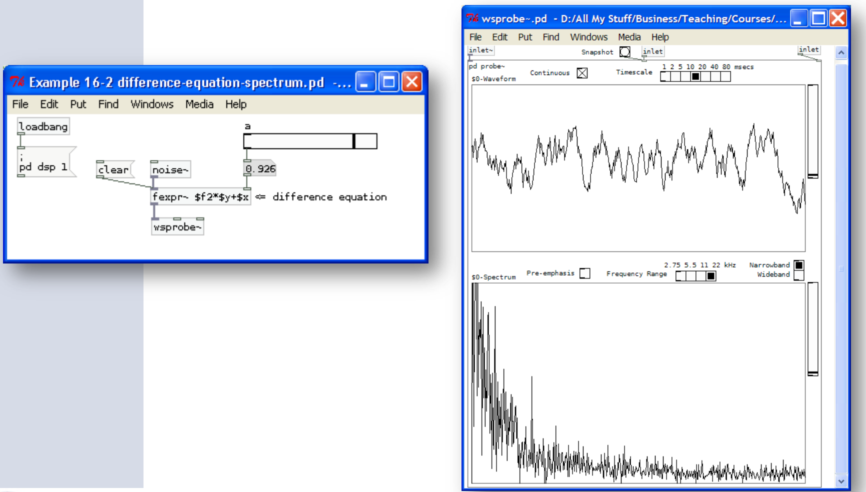
26

Filter Response: *real pole*



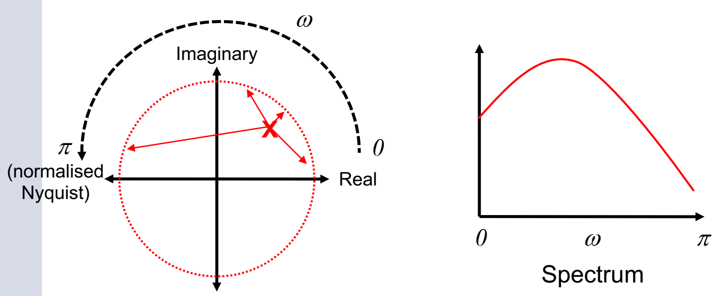
27

Filter Response: *real pole*



28

Filter Response: *complex pole*



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29

Poles & Zeros in Pd

- As well as [fexpr~], Pd provides several 'raw' pole/zero filter objects:
  - real pole [rpole~]
  - real zero [rzero~]
  - complex pole [cpole~]
  - complex zero [czero~]
- Each of these objects takes the appropriate real/complex filter coefficients as inputs

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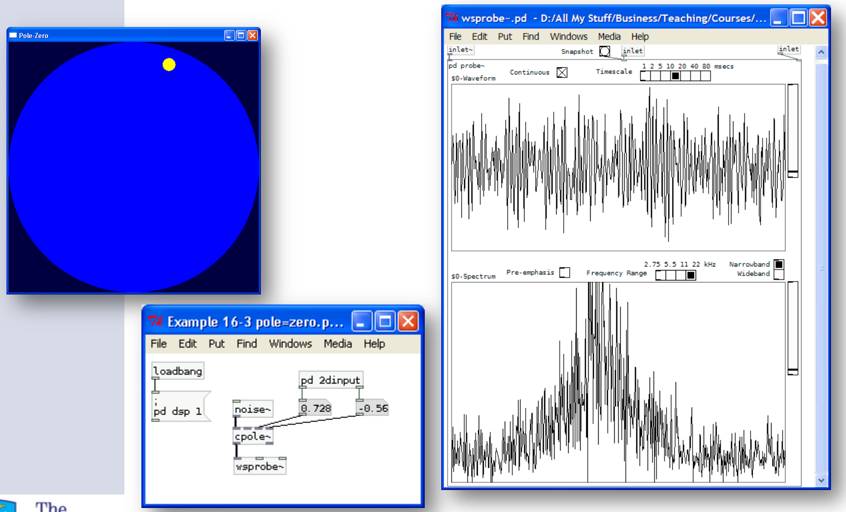
30

Filter Response: *complex pole*



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31

This lecture has covered …

- Filtering
- Difference equations
- Convolution
- The Z transform
- Link with the Fourier transform
- Poles and zeros
- The Z Plane

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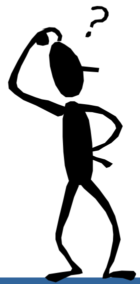
32

Any Questions ?



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Next time …

Linear Filters

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34