

Adjunctions

Consider preorders (A, \leq) - reflexive, transitive

Eg (\mathbb{R}, \leq) ; $(\mathbb{P}A, \subseteq)$; $(\mathbb{Z}^*, \leq\#)$

A monotonic function $f: (A, \leq_A) \rightarrow (B, \leq_B)$ is $f: A \rightarrow B$ such that $a \leq_A a' \Rightarrow f a \leq_B f a'$.

A Galois connection

between preorders $(A, \leq_A) \xleftarrow{g} (B, \leq_B) \xrightarrow{f}$ is monotonic f, g such that

$$f a \leq_B b \Leftrightarrow a \leq_A g b$$

Eg

$$\text{real } n \leq_{\mathbb{R}} x \Leftrightarrow n \leq_{\mathbb{Z}} \text{floor } x \quad (\mathbb{R}, \leq_{\mathbb{R}}) \xleftarrow{\text{real}} (\mathbb{Z}, \leq_{\mathbb{Z}}) \xrightarrow{\text{floor}}$$

Proofs often simplified by this transposition, eg $\text{real}(\text{floor } x) \leq_{\mathbb{R}} x \leq \text{real}(\text{floor } x + 1)$; div.

Generalizes inverses ($g = f^{-1}$).

Say f is left or lower adjoint of g , and g is right or upper adjoint of f .

Each adjoint determines the other:

$f a$ is least b st $a \leq_A g b$, ~~if~~

$g b$ is greatest a st $f a \leq_B b$

Composited $gf = g \circ f: A \rightarrow A$ a closure:

$$a \leq_A gf a, \quad gf(gf a) \leq_A gf a$$

Let's categorify!

Preorder (A, \leq) induces category with objects A , unique arrow $a \rightarrow b \Leftrightarrow a \leq b$.
 Identity, composition \sim reflexivity, transitivity
 Functor \sim monotonic function

$$f: a \rightarrow b \Leftrightarrow a \leq b \Rightarrow fa \leq fb \Leftrightarrow Ff: fa \rightarrow fb$$

$$\text{id}, F\text{id}: fa \rightarrow fa \text{ so equal, sim } Fg \circ Ff$$

Galois connection \rightsquigarrow adjunction $L \rightarrow R$
 between functors

$$L: \mathcal{D} \rightarrow \mathcal{C} \text{ and } R: \mathcal{C} \rightarrow \mathcal{D}$$

such that there is isomorphism

$[_] : \mathcal{C}(LA, B) \cong \mathcal{D}(A, RB)$: Γ natural: A, B .
 L, R are left, right adjoint. $[_], \Gamma$ adjuncts

Equivalence $f = \Gamma g \Leftrightarrow Lf = g$. Naturality

$$Rk \cdot Lf \cdot h = Lk \cdot f \cdot Lh$$

$$k \cdot \Gamma g \cdot Lh = \Gamma Rk \cdot g \cdot h$$

Unit $\eta_A = [\text{id}_{LA}] : 1 \rightarrow RL$, $\varepsilon_B = [\text{id}_{RB}] : LR \rightarrow 1$
 determine adjoints, by naturality.

Left adjoint preserves initial objects:

$$L \emptyset \rightarrow B \cong \emptyset \rightarrow RB$$

(more generally, preserve all colimits).

Eg (co-) product. Product category $\mathcal{C} \times \mathcal{D}$ has object pairs (X, Y) , arrow pairs $(f: \mathcal{C}(X, Y), g: \mathcal{D}(X, Y))$.

Diagonal functor

$$\Delta: \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C} \text{ has } \mathcal{C} \xleftarrow{(+)} \mathcal{C} \times \mathcal{C} \xleftarrow{\Delta} \mathcal{C}$$

$$\Delta A = (A, A), \Delta f = (f, f).$$

Two adjunctions $(+) \dashv \Delta \dashv (-)$.

Adjunct $L(f, g): \Delta A \rightarrow (B, C) \dashv \vdash A \rightarrow B \times C$ write $f \Delta g$,
 adj $\Gamma(f, g): (A, B) \rightarrow \Delta C \dashv \vdash A + B \rightarrow C$ write $f \nabla g$.

Eg currying in CCC:

$$\mathcal{C}(A \times P, B) \cong \mathcal{C}(A, B^P). \quad \mathcal{C} \xleftarrow{-xP} \mathcal{C}$$

$\lfloor f: A \times P \rightarrow B \rfloor: A \rightarrow B^P = \text{curry } f$,

$\lfloor g: A \rightarrow B^P \rfloor: A \times P \rightarrow B = \text{uncurry } g$.

Eg free (co-) algebras

$F\text{-Alg}(\mathcal{C})$ has

objects

F -algebras

$$F\text{-Alg}(\mathcal{C}) \xleftarrow{\text{Free}^F} \mathcal{C} \xleftarrow{u_g} \mathcal{C}\text{-Coalg}(\mathcal{C})$$

$$\xrightarrow{u^F} \quad \xrightarrow{\text{Cofree}_g}$$

$(A, a: FA \rightarrow A)$, arrows $h: (A, a) \rightarrow (B, b)$ st $h \circ a = b \circ h$.

$\text{Free}^F A = (\mu X. A + FX, \text{In})$ terms with variables (of A).

$\text{Free}^F(f: A \rightarrow B)$ renames variables. $\text{Free}^F \emptyset \cong \mu^F$

$u^F(A, a) = A, u^F h = h$.

$F\text{-Alg}(\mathcal{C})(\text{Free}^F A, (B, b)) \cong \mathcal{C}(A, u^F(B, b))$.

Dually, $\text{Cofree}_g A = (\nu X. A \times GX, \text{out}) - \mathcal{C}$ branching.

Mutunorphism

$$h \circ In = f \circ F(h \circ k)$$

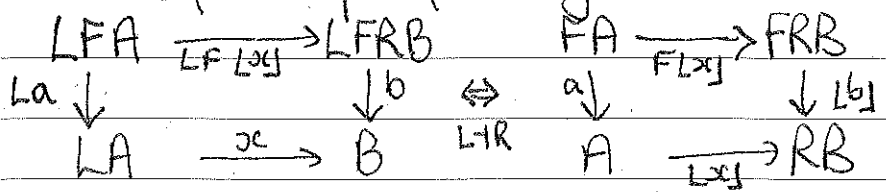
$$k \circ In = g \circ F(h \circ k)$$

eg perfect :: Tree a -> Bool, depth :: Tree a -> Int

Pairing: $h \circ In = f \circ F(h \circ k) \wedge k \circ In = \dots$

$$\Leftrightarrow (h, k) \circ \Delta In = (f, g) \circ \Delta FL(h, k)$$

Abstract from specific adjunction $\Delta \dashv X$



$$ie \alpha \circ La = b \circ LF \lfloor \alpha \rfloor \Leftrightarrow \lfloor \alpha \rfloor \circ a = \lfloor b \rfloor \circ F \lfloor \alpha \rfloor$$

Specialise (A, a) to (In, In):

$$\Leftrightarrow \alpha \circ LIn = b \circ LF \lfloor \alpha \rfloor$$

$$\Leftrightarrow \lfloor \alpha \rfloor = fold_F \lfloor b \rfloor$$

$$\Leftrightarrow \alpha = \lceil fold_F \lfloor b \rfloor \rceil$$

for $\Delta \dashv X$, $\lfloor (h, k) \rfloor = h \circ k$ and $\lceil f \rceil = (fst \circ f, snd \circ f)$

Accumulating fold $h (Int) p = f (F h) p$

eg depths :: Tree a -> Int

$$\Leftrightarrow \forall t, p: \alpha (Int, p) = f (F (curry \alpha) t, p)$$

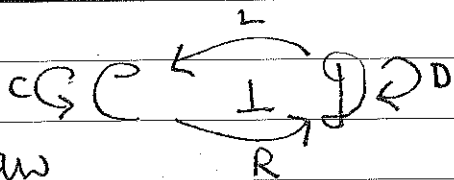
$$\Leftrightarrow \alpha \circ In \times id_p = b \circ F (curry \alpha) \times id_p$$

$$\Leftrightarrow \alpha \circ (X P) In = b \circ (X P) F \lfloor \alpha \rfloor$$

$$\Leftrightarrow \lfloor \alpha \rfloor = fold_F \lfloor b \rfloor$$

$$\Leftrightarrow \alpha = uncurry (fold_F (curry b))$$

More generally,
one has situation
with distributive law



$\sigma: LD \rightarrow CL$. Its conjugate is $\tau: DR \rightarrow RC$
satisfying $[Cf \circ \sigma_A] = [\tau_B \circ D[f]]$ for $f: LA \rightarrow B$

In fact, define $\tau_B = [C\varepsilon_B \circ \sigma_{RB}]$.

Adjoint fold equation for $b: CB \rightarrow B$

$$\alpha \circ L \text{In} = b \circ C\alpha \circ \sigma_{\mu D} : LD\mu D \rightarrow B$$

Now $\alpha \circ La = b \circ C\alpha \circ \sigma_A$

\Leftrightarrow $\llbracket L \dashv \text{an isomorphism} \rrbracket$

$$L\alpha \circ La = [b \circ C\alpha \circ \sigma_A]$$

\Leftrightarrow $\llbracket \text{naturality} \rrbracket$

$$L\alpha \circ a = Rb \circ L[C\alpha \circ \sigma_A]$$

\Leftrightarrow $\llbracket \text{conjugate } \sigma \dashv \tau \rrbracket$

$$L\alpha \circ a = Rb \circ \tau_B \circ D[L\alpha]$$

In particular, for $(A, a) = (\mu D, \text{In})$:

$$\alpha \circ L \text{In} = b \circ C\alpha \circ \sigma_{\mu D} \Leftrightarrow \alpha = \llbracket \text{fold}_D (Rb \circ \tau_B) \rrbracket$$

Special case for "canonical" $C = LDR$ has

$$\sigma = LD\eta : LD \rightarrow LDRL \quad \tau = \eta DR : DR \rightarrow RLDR$$

then $C\alpha \circ \sigma_A = LD[L\alpha]$, $Rb \circ \tau_B = [b]$.