Adjunctions

Consider preorders $(A, \leq)$ – reflexive, transitive.

Eg $(\mathbb{R}, \leq)$; $(\mathbb{P} A, \subseteq)$; $(\Sigma^*, \leq_\#)$

A <u>monotonic function</u> $f : (A, \leq_A) \to (B, \leq_B)$ is $f : A \to B$ such that $a \leq_A a' \Rightarrow f a \leq_B f a'$.

A <u>Galois connection</u> between preorders $(A, \leq_A) \underset{f}{\overset{g}{\leftrightarrows}} (B, \leq_B)$ is monotonic f, g such that

$$f\,a \leq_B b \iff a \leq_A g\,b$$

Eg

$\text{real}\ n \leq_{\mathbb{R}} x \Leftrightarrow n \leq_{\mathbb{Z}} \text{floor}\ x$ $\quad (\mathbb{R}, \leq_{\mathbb{R}}) \underset{\text{floor}}{\overset{\text{real}}{\leftrightarrows}} (\mathbb{Z}, \leq_{\mathbb{Z}})$

Proofs often simplified by this transposition, eg $\text{real}(\text{floor}\ x) \leq_{\mathbb{R}} x < \text{real}(\text{floor}\ x + 1)$; div.

Generalizes inverses $(g = f^{-1})$.

Say f is <u>left</u> or <u>lower adjoint</u> of g, and g is <u>right</u> or <u>upper adjoint</u> of f.

Each adjoint determines the other:

$f\,a$ is least b st $a \leq_A g\,b$,

$g\,b$ is greatest a st $f\,a \leq_B b$

Composite $gf = g \circ f : A \to A$ a <u>closure</u>:

$a \leq_A gf\,a$, $gf\,(gf\,a) \leq_A gf\,a$

Let's categorify!

Preorder $(A, \leq)$ induces category with objects A, unique arrow $a \to b \Leftrightarrow a \leq b$.
Identity, composition ~ reflexivity, transitivity
Functor ~ monotonic function

$$f : a \to b \Leftrightarrow a \leq b \Rightarrow Fa \leq Fb \Leftrightarrow Ff : Fa \to Fb$$

$$id, F\,id : Fa \to Fa \text{ so equal, sim } Fg \circ Ff$$

Galois connection ⇝ adjunction $L \dashv R$ between functors

$L : \mathcal{D} \to \mathcal{C}$ and $R : \mathcal{C} \to \mathcal{D}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend right, "R"'] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend right, "L"'] \end{tikzcd}$$

such that there is isomorphism

$$\lfloor - \rfloor : \mathcal{C}(LA, B) \cong \mathcal{D}(A, RB) : \lceil - \rceil \text{ natural in } A, B.$$

L, R are left, right adjoint. $\lfloor - \rfloor, \lceil - \rceil$ adjuncts.
Equivalence $f = \lceil g \rceil \Leftrightarrow \lfloor f \rfloor = g$. Naturality

$$Rk \cdot \lfloor f \rfloor \cdot h = \lfloor k \cdot f \cdot Lh \rfloor$$

$$k \cdot \lceil g \rceil \cdot Lh = \lceil Rk \cdot g \cdot h \rceil$$

Unit $\eta_A = \lfloor id_{LA} \rfloor : 1 \dot{\to} RL$, $\varepsilon_B = \lceil id_{RB} \rceil : LR \dot{\to} 1$
determine adjuncts, by naturality.
Left adjoint preserves initial objects:

$$L0 \to B \cong 0 \to RB$$

(more generally, preserve all colimits).

Eg (co-)product. Product category $\mathcal{C}\times\mathcal{D}$ has object pairs (X,Y), arrow pairs $(f:\mathcal{C}(X,Y), g:\mathcal{D}(X',Y'))$.
Diagonal functor
$\Delta:\mathcal{C}\to\mathcal{C}\times\mathcal{C}$ has
$\Delta A=(A,A)$, $\Delta f=(f,f)$.

$$\begin{tikzcd} \mathcal{C} \arrow[r, phantom, "\perp"] \arrow[r, bend right, "\Delta"'] & \mathcal{C}\times\mathcal{C} \arrow[l, bend right, "(+)"'] \arrow[r, phantom, "\perp"] \arrow[r, bend right, "(\times)"'] & \mathcal{C} \arrow[l, bend right, "\Delta"'] \end{tikzcd}$$

Two adjunctions $(+)\dashv\Delta\dashv(\times)$.
Adjunct $\lfloor (f,g):\Delta A\to(B,C)\rfloor : A\to B\times C$ write $f\vartriangle g$,
ad $\lceil (f,g):(A,B)\to\Delta C\rceil : A+B\to C$ written $f\triangledown g$.
Eg currying in CCC:
$\mathcal{C}(A\times P, B)\cong\mathcal{C}(A,B^P)$.

$$\begin{tikzcd} \mathcal{C} \arrow[r, phantom, "\perp"] \arrow[r, bend right, "(-)^P"'] & \mathcal{C} \arrow[l, bend right, "-\times P"'] \end{tikzcd}$$

$\lfloor f:A\times P\to B\rfloor : A\to B^P = \text{curry}\, f$,
$\lceil g:A\to B^P\rceil : A\times P\to B = \text{uncurry}\, g$.
Eg free (co-)algebras
F-Alg($\mathcal{C}$) has objects F-algebras

$$\begin{tikzcd} F\text{-Alg}(\mathcal{C}) \arrow[r, phantom, "\perp"] \arrow[r, bend right, "U^F"'] & \mathcal{C} \arrow[l, bend right, "\mathrm{Free}^F"'] \arrow[r, phantom, "\perp"] \arrow[r, bend right, "\mathrm{Cofree}_G"'] & G\text{-Coalg}(\mathcal{C}) \arrow[l, bend right, "U_G"'] \end{tikzcd}$$

$(A, a:FA\to A)$, arrows $h:(A,a)\to(B,b)$ st $h\circ a=b\circ Fh$.
$\mathrm{Free}^F A = (\mu X.\, A+FX, \mathrm{In})$ terms with variables $\in A$.
$\mathrm{Free}^F(f:A\to B)$ renames variables. $\mathrm{Free}^F\emptyset \cong \mu F$
$U^F(A,a)=A$, $U^F h = h$.
$F\text{-Alg}(\mathcal{C})(\mathrm{Free}^F A, (B,b)) \cong \mathcal{C}(A, U^F(B,b))$.
Dually, $\mathrm{Cofree}_G A = (\nu X.\, A\times GX, \mathrm{out})$ - G branching.

Mutumorphism $h \circ In = f \circ F(h \vartriangle k)$

$k \circ In = g \circ F(h \vartriangle k)$

eg $perfect :: Tree\ a \to Bool$, $depth :: Tree\ a \to Int$

Pairing: $h \circ In = f \circ F(h \vartriangle k) \wedge k \circ In = \ldots$

$\Leftrightarrow (h,k) \circ \Delta In = (f,g) \circ \Delta F \lfloor (h,k) \rfloor$

Abstract from specific adjunction $\Delta \dashv \times$

$$
\begin{tikzcd}
LFA \arrow[r, "LF\lfloor x \rfloor"'] \arrow[d, "La"'] & LFRB \arrow[d, "b"] \arrow[r, phantom, "\overset{\Leftrightarrow}{L \dashv R}"] & FA \arrow[r, "F\lfloor x \rfloor"'] \arrow[d, "a"'] & FRB \arrow[d, "\lfloor b \rfloor"] \\
LA \arrow[r, "x"] & B & A \arrow[r, "\lfloor x \rfloor"'] & RB
\end{tikzcd}
$$

ie $x \circ La = b \circ LF\lfloor x \rfloor \Leftrightarrow \lfloor x \rfloor \circ a = \lfloor b \rfloor \circ F\lfloor x \rfloor$

Specialise (A, a) to $(\mu F, In)$:

$x \circ L\,In = b \circ LF\lfloor x \rfloor$

$\Leftrightarrow \lfloor x \rfloor = fold_F \lfloor b \rfloor$

$\Leftrightarrow x = \lceil fold_F \lfloor b \rfloor \rceil$

For $\Delta \dashv \times$, $\lfloor (h,k) \rfloor = h \vartriangle k$ and $\lceil f \rceil = (fst \circ f, snd \circ f)$

Accumulating fold $h\,(In\ t)\ p = f\,(F\,h\,t)\ p$

eg $depths :: Tree\ a \to Tree\ Int$

$\forall t, p : x(In\ t, p) = b(F(curry\ x)\ t, p)$

$\Leftrightarrow x \circ In \times id_P = b \circ F(curry\ x) \times id_P$

$\Leftrightarrow x \circ (\times P) In = b \circ (\times P) F \lfloor x \rfloor$

$\Leftrightarrow \lfloor x \rfloor = fold_F \lfloor b \rfloor$

$\Leftrightarrow x = uncurry\,(fold_F\,(curry\ b))$

More generally, one has situation
$$\begin{tikzcd} \mathcal{C} \arrow[loop left, "C"] \arrow[r, bend right, "R"'] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend right, "L"'] \arrow[loop right, "D"] \end{tikzcd}$$
with distributive law $\sigma : LD \xrightarrow{\cdot} CL$. Its conjugate is $\tau : DR \xrightarrow{\cdot} RC$ satisfying $\lfloor Cf \circ \sigma_A \rfloor = \tau_B \circ D\lfloor f \rfloor$ for $f : LA \to B$.
In fact, define $\tau_B = \lfloor C\varepsilon_B \circ \sigma_{RB} \rfloor$.

<u>Adjoint fold</u> equation for $b : CB \to B$

$$x \circ L\,In = b \circ Cx \circ \sigma_{\mu D} \;:\; LD\mu D \to B.$$

Now $x \circ La = b \circ Cx \circ \sigma_A$

$\Leftrightarrow$ $[\![\ \lfloor - \rfloor$ an isomorphism $]\!]$

$\lfloor x \circ La \rfloor = \lfloor b \circ Cx \circ \sigma_A \rfloor$

$\Leftrightarrow$ $[\![$ naturality $]\!]$

$\lfloor x \rfloor \circ a = Rb \circ \lfloor Cx \circ \sigma_A \rfloor$

$\Leftrightarrow$ $[\![$ conjugate $\sigma \dashv \tau$ $]\!]$

$\lfloor x \rfloor \circ a = Rb \circ \tau_B \circ D\lfloor x \rfloor$

In particular, for $(A, a) = (\mu D, In)$:

$$x \circ L\,In = b \circ Cx \circ \sigma_{\mu D} \Leftrightarrow x = \lceil fold_D\,(Rb \circ \tau_B) \rceil$$

Special case for "canonical" $C = LDR$ has

$\sigma = LD\eta : LD \xrightarrow{\cdot} LDRL$ + $\tau = \eta DR : DR \xrightarrow{\cdot} RLDR$

then $Cx \circ \sigma_A = LD\lfloor x \rfloor$, $Rb \circ \tau_B = \lfloor b \rfloor$.