Monads

Monad (M, η, μ) is functor $M : \mathcal{C} \to \mathcal{C}$, multiplicative $\mu : MM \dot\to M$ and unit $\eta : 1 \dot\to M$ such that

$$\begin{tikzcd} MMM \arrow[r, "M\mu"] \arrow[d, "\mu M"'] & MM \arrow[d, "\mu"] \\ MM \arrow[r, "\mu"] & M \end{tikzcd} \qquad \begin{tikzcd} 1M \arrow[r, "\eta M"] \arrow[dr, equal] & MM \arrow[d, "\mu"] & M1 \arrow[l, "M\eta"'] \arrow[dl, equal] \\ & M & \end{tikzcd}$$

Eg lists (List, single, concat)
Eg option (Maybe, Just, μ) where...
Eg state $((- \times S)^S, \eta, \mu)$ where...
Eg free $((\mu X. - + FX), \text{var}, \text{sub})$ where...
Useful for expressing effects in FP.

Kleisli category $\mathcal{C}_M$ has same object as $\mathcal{C}$, arrows $\mathcal{C}_M(A,B) = \mathcal{C}(A, MB)$, identity $id_A = \eta_A \in \mathcal{C}_M(A,A)$, composition $g \bullet f = \mu \circ Mg \circ f$.

Adjunction $\hat{(-)} \dashv U$ where

$$\begin{tikzcd} \mathcal{C}_M \arrow[r, bend left, "U"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "\hat{(-)}"] \end{tikzcd}$$

$\hat{A} = A$, $\widehat{(f : A \to B)} = \eta_B \circ f$,
$UA = MA$, $U(g : A \to_M B) = \mu_B \circ Mg$

Adjuncts $\lfloor - \rfloor : \mathcal{C}_M(\hat{A}, B) \cong \mathcal{C}(A, UB) : \lceil - \rceil$ identities.

Unit $\eta_A = \lfloor id_{\hat{A}} \rfloor = \eta_A$, counit $\varepsilon_B = \lceil id_{MB} \rceil = id_{MB}$.

$U\hat{(-)} = M$ — $U\hat{A} = MA$, $U\hat{f} = Mf$. Indeed, $L \dashv R \Rightarrow (RL, \eta, R\varepsilon L)$ a monad. (Converse?)

Monads on preorders are closure ops.

Lifting $F : \mathcal{C} \to \mathcal{C}$ to $F_M : \mathcal{C}_M \to \mathcal{C}_M$.
Try $F_M A = FA$. Then $F_M(f : A \to_M B)$
$: F_M A \to_M F_M B = FA \to MFB$. Now $Ff : FA \to FMB$
so want distributive law $\delta : FM \Rightarrow MF$
in $\mathcal{C}$; then define $F_M f = [\delta_B \circ F\lfloor f \rfloor]$.
Functor laws: identity

$$\left.\begin{aligned} &F_M(id_A : A \to_M A) = \delta_A \circ F\eta_A \\ &id_{F_M A} = \eta_{FA} \end{aligned}\right\} \text{should agree}$$

and composition of $f : A \to_M B$ and $g : B \to_M C$

$$\left.\begin{aligned} F_M(g \bullet f) &= \delta_C \circ F\mu_C \circ FMg \circ Ff \\ F_M g \bullet F_M f &= \mu_C \circ M(\delta_C \circ Fg) \circ \delta_B \circ Ff \\ &= \mu_C \circ M\delta_C \circ \delta_C \circ FMg \circ Ff \end{aligned}\right\}$$

A <u>distributive law of F over M</u> is $\delta : FM \Rightarrow MF$
st

$$\delta \circ F\eta = \eta$$
$$\delta \circ F\mu = \mu \circ M\delta \circ \delta$$

(sometimes called "M over F"...).
Then define $\hat{F}$ to be F_M.

F-algebras lift too. By defn F_m-Alg$(\mathcal{C}_M)$ has objects $(A, a : F_M A \to_m A)$, arrows $h : A \to_m B$ st $h \bullet a = b \bullet F_m h$.

When $F_m = \hat{F}$ a lifting, everything else lifts.

$$(A,a) \in F\text{-Alg}(\mathcal{C}) \Rightarrow (\hat{A}, \hat{a} : \hat{F}\hat{A} \to_m \hat{A}) =$$
$$(A, \eta_A \circ a : FA \to MA) \in \hat{F}\text{-Alg}(\mathcal{C}_m).$$

$$h : (A,a) \to (B,b) \Rightarrow \hat{h} : \hat{A} \to_m \hat{B} \text{ and}$$
$$\hat{h} \bullet \hat{a} = \widehat{h \circ a} = \widehat{b \circ Fh} = \hat{b} \bullet \hat{F}h = \hat{b} \bullet \hat{F}\hat{h}$$

Moreover, lifting respects identity, comp.

So $\hat{(-)}$ a functor $F\text{-Alg}(\mathcal{C}) \to \hat{F}\text{-Alg}(\mathcal{C}_m)$.

Forgetful functor

$V : \hat{F}\text{-Alg}(\mathcal{C}_m) \to F\text{-Alg}(\mathcal{C})$ given by

$$\begin{tikzcd} \hat{F}\text{-Alg}(\mathcal{C}_m) \arrow[r, bend right, "V"'] \arrow[r, phantom, "\perp"] & F\text{-Alg}(\mathcal{C}) \arrow[l, bend right, "\hat{(-)}"'] \end{tikzcd}$$

$$V(A, a : \hat{F}A \to_m A) = (MA, \mu \circ Ma \circ \delta)$$
$$V(h : (A,a) \to_m (B,b)) = \mu \circ Mh$$

with $\hat{(-)} \dashv V$.

Since $\hat{(-)}$ a left adjoint, preserves initial objects. So if $(\mu F, In)$ initial in F-Alg$(\mathcal{C})$ then $\widehat{(\mu F, In)} = (\mu F, \eta_{\mu F} \circ In)$ initial in $\hat{F}$-Alg$(\mathcal{C}_m)$.

Write $mfold_F(a : FA \to_m A) : \mu F \to_m A$.

UP $h = mfold_F\, a \Leftrightarrow h \bullet (\eta \circ In) = a \bullet \hat{F}h$

$$\Leftrightarrow h \circ In = \mu_A \circ Ma \circ \delta_A \circ Fh$$

Previous construction requires
$\delta : FM \xrightarrow{\cdot} MF$ coherent with η, μ.
Straightforward to define for some F:
constants, sums, compositions.
But two obvious definitions for product
liftM2 (g) flip (liftM2 (flip (g)))
neither coherent with μ in general.
Coherence regained if M commutative
(when two product definitions coincide).
Proper δ exists for polynomial F
that don't use "full" products,
only eg (Ax).

If δ is not coherent, but still right type, can still define F_M and $mfold_F$ as before. But F_M not a functor, so we don't get lifting or initiality. The universal property doesn't hold; in particular, no fusion

$$h \bullet mfold_F\, a = mfold_F\, b$$
$$\Leftarrow h \circ a = b \bullet \hat{F} h$$

So monadic folds of limited use. ☹
They are, however, adjoint folds. ☺