Undecidability and the Halting Problem, Countability and Uncountability, Unrecognisability – Lecture 15

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The Halting Problem

Theorem

The language $A_{TM} = \{\langle M,w\rangle | M \text{ is a TM and } M \text{ accepts } w\}$ is undecidable

Proof Sketch (by Contradiction)

Assume A_{TM} is <u>decidable</u> and H is a decider for it, so

$$H(\langle M,w\rangle) = \begin{cases} \text{accept} & \text{if} \\ \\ \text{reject} & \text{if} \end{cases}$$

Now construct a new TM D that uses H as a subroutine

$$D(\langle M\rangle) = \begin{cases} \text{accept} & \text{if } H \text{ says} \\ \\ \text{reject} & \text{if } H \text{ says} \end{cases}$$

What happens when D is run with $\langle D\rangle$ as input?

$$D(\langle D\rangle) = \begin{cases} \text{accept} & \text{if } H \text{ says} \\ \\ \text{reject} & \text{if } H \text{ says} \end{cases}$$

Definition

A set is *countable* if it is finite, or it can be put in a one-to-one correspondence with the natural numbers

Examples

Even numbers

Rational numbers

Definition

A set is *uncountable* if it is infinite and cannot be put into a one-to-one correspondence with the natural numbers

Theorem

The set of real numbers is uncountable

Proof Sketch (using *Diagonalisation*)

Theorem

Some languages are not Turing-recognisable

Proof Sketch (using Diagonalisation)

Language Hierarchy